ON THE MINIMUM SEMIDEFINITE RANK OF A SIMPLE GRAPH

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Abstract. The minimum semidefinite rank of a graph is defined to be the minimum rank among all positive semidefinite matrices whose zero/nonzero pattern corresponds to that graph. We recall some known facts and present new results, including results concerning the effects of vertex or edge removal from a graph on minimum semidefinite rank.

1. Preliminaries

1.1. **Introduction.** A *graph* is a pair $G = (V(G), E(G))$ where $V(G)$ is the finite, nonempty set of vertices and $E(G)$ is the set of edges, which are unordered pairs of vertices. A graph is *simple* if it has no multiple edges or loops. In this paper, we will study only simple graphs. The *order* of G, denoted by $|G|$, is the cardinality of $V(G)$.

The set of n by n matrices with entries that are complex numbers is denoted by $M_n(\mathbb{C})$. A matrix $A \in M_n(\mathbb{C})$ is Hermitian (or self-adjoint) if A equals its conjugate transpose. Given a Hermitian matrix $A \in M_n(\mathbb{C})$, the graph of A is the graph on n vertices $1, \ldots, n$ that has an edge between vertices i and j (where $i \neq j$) if and only if the i,jth entry of A is nonzero. By definition, this graph is independent of the real diagonal entries of A. The set of all complex Hermitian matrices that share a common graph G is denoted by $\mathcal{H}(G)$. A matrix $A \in M_n(\mathbb{C})$ is defined to be *positive semidefinite* (psd) if A is Hermitian and $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$. Given a graph G, denote by $\mathcal{P}(G)$ the set of all psd matrices with graph G.

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If $A \in M_n(\mathbb{C})$ and α and β are index sets contained in $S = \{1, \dots, n\}$, we denote by $A[\alpha, \beta]$ the submatrix of A whose rows are indexed by α and whose columns are indexed by β. We write the principal submatrix $A[\alpha, \alpha]$ as $A[\alpha]$, $A[S \setminus \alpha, S \setminus \beta]$ as $A(\alpha, \beta)$, and $A(\alpha, \alpha)$ as $A(\alpha)$. We will denote the column space of A by $\mathrm{col}(A)$, the row space by $\mathrm{row}(A)$, and the null space by $\mathrm{null}(A)$.

In 1996, Nylen [15] studied the minimum rank among real symmetric matrices with a given graph, giving rise to the area of *minimum rank problems*. The real symmetric problem, as well as the corresponding one for matrices in $\mathcal{H}(G)$, have since been considered by many others [1,2,5,6,8]. In this paper, we consider the *minimum semidefinite rank problem*, which seeks to determine the *minimum semidefinite rank* (msr) of a graph G, $\mathrm{msr}(G)$, the minimum rank among matrices in $\mathcal{P}(G)$. We now recall some interesting and useful results concerning msr.

The Laplacian matrix of a connected graph G of order n (see Merris [14] for a survey) is psd and has rank equal to $n-1$, which implies that $\mathrm{msr}(G) \leq n-1$ for every connected graph G. In fact, if G is a connected graph of order n, then $\mathrm{msr}(G) = n-1$ if and only if G is a tree [12,17]. On the other hand, if we consider the complete graph K_n of order $n > 1$ (a *clique* of cardinality n), the n by n matrix of all ones is psd, which shows that $\mathrm{msr}(K_n) = 1$. Further, $\mathrm{msr}(G) = 1$ if and only if $G = K_n$ with $n > 1$. Thus for a connected graph of order $n > 1$, $1 \leq \mathrm{msr}(G) \leq n-1$. Note that $\mathrm{msr}(K_1) = 0$.

Induced subgraphs of a connected graph $G = (V(G), E(G))$ provide useful lower bounds for $\mathrm{msr}(G)$. Recall that the subgraph H of G induced by $R \subseteq V(G)$ is the subgraph of G with vertex set R and edge set consisting of those edges of G where both vertices are elements of R. For any index set α, if $A \in M_n(\mathbb{C})$ is psd, then $A[\alpha]$ is psd and $\mathrm{rank}\, A \geq \mathrm{rank}\, A[\alpha]$ [10, p. 397]. It follows that if H is an induced subgraph of G, $\mathrm{msr}(G) \geq \mathrm{msr}(H)$. For one example, let $\mathrm{ts}(G)$ be the *tree size* [7] of G, the number of vertices in a maximum induced tree. The results mentioned so far show that if G is a connected graph of order at least two, then $\mathrm{msr}(G) \geq \mathrm{ts}(G) - 1$.

A set of vertices is said to be *independent* if they are pairwise disjoint. The *independence number* $\alpha(G)$ of a graph G is the cardinality of the largest independent set of vertices in $V(G)$. It was shown in [4] that, for a connected graph G, $\mathrm{msr}(G) \geq \alpha(G)$. Let C_n be a cycle on n vertices. It was shown by van der Holst [17] that $\mathrm{msr}(C_n) = n-2$. Thus $\mathrm{msr}(G) - \alpha(G)$ may be arbitrarily large.

For a vertex w of a graph G, let $N(w)$ denote the set of all vertices adjacent to w in G, called the *neighborhood* of w in G. By the *closed* neighborhood of w, $\overline{N}(w)$, we mean $w \cup N(w)$.

Recall that a graph is *chordal* if it does not have an induced subgraph that is a cycle on four or more vertices, and a vertex is *simplicial* if its neighborhood is a clique. It is well known that every chordal graph has a simplicial vertex [18].

Finally, since the direct sum of the matrices for the connected components of a graph is a matrix for the entire graph and the rank of the direct sum is equal to the sum of the ranks of its direct summands, we assume, whenever possible, graphs to be connected.

1.2. **Vector Representations.** If $\vec{x}$ and $\vec{y}$ are column vectors in $\mathbb{C}^n$, the *inner product* of $\vec{x}$ and $\vec{y}$ is the complex number $\vec{y}^*\vec{x}$ and is denoted by $\langle \vec{x}, \vec{y} \rangle$. The vectors $\vec{x}$ and $\vec{y}$ are *orthogonal* if $\langle \vec{x}, \vec{y} \rangle = 0$. We will denote by $\vec{x}^\perp$ the subspace of $\mathbb{C}^n$ orthogonal to $\vec{x}$. If $\vec{x} \neq \vec{0}$, then $\vec{x}^\perp$ has dimension $n-1$.

Given n complex column vectors $\vec{X} = \{\vec{x}_1, \ldots, \vec{x}_n\}$, where each $\vec{x}_i \in \mathbb{C}^m$, $1 \leq i \leq n$, the matrix

$$\begin{bmatrix} \vec{x}_1^* \\ \vec{x}_2^* \\ \vdots \\ \vec{x}_n^* \end{bmatrix} \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \ldots & \vec{x}_n \end{bmatrix}$$

is a psd matrix called the *Gram matrix* of $\vec{X}$. Its associated graph G has n vertices $\{v_1, \ldots, v_n\}$ corresponding to the vectors $\{\vec{x}_1, \ldots, \vec{x}_n\}$, and edges corresponding to nonzero inner products among those vectors. Consequently, $\vec{X}$ is called a *vector representation* of G.

Conversely, any psd matrix A may be factored (far from uniquely) as Y^*Y with $\operatorname{rank} A = \operatorname{rank} Y$, and so A is a Gram matrix. By $\operatorname{rank} \vec{X}$, we mean the dimension of the span of the vectors in $\vec{X}$, which is equal to $\operatorname{rank} X^*X$ [10, Theorem 7.2.10]. Thus finding a psd matrix with a given graph and finding a vector representation of the graph are equivalent problems.

Two vertices u and v of G are *duplicate* vertices if $\overline{N}(u) = \overline{N}(v)$. Further, define $D(v) = V(G) \setminus \overline{N}(v)$. We denote the induced subgraph of G resulting from the deletion of a vertex v by $G - v$. Since duplicate vertices may be represented by the same vector, removing a duplicate vertex does not affect the minimum semidefinite rank [4, Corollary 2.3]. This affords another proof that $\operatorname{msr}(K_n) = 1$ for $n \geq 2$.

Recall that the *degree* of a vertex v in G, $d_G(v)$, is the cardinality of $N(v)$. If $d_G(v) = 1$, then v is said to be a *pendant* vertex.

A vector representation of a graph may be sometimes constructed from a vector representation of an induced subgraph. We say a graph G is the *superposition* of two graphs G_1 and G_2 if G may be obtained by identifying G_1 and G_2 at a set of vertices, keeping all edges that are present in either G_1 or G_2.

Theorem 1.1 ([4]). *Let G be the superposition of G_1 and G_2. Further suppose $\{\vec{x}_i\}$ and $\{\vec{w}_i\} \subseteq \mathbb{C}^n$ are vector representations of G_1 and G_2 such that $(v_i, v_j) \notin E$ for $i \neq j$ implies $\langle \vec{x}_i, \vec{w}_j \rangle = \langle \vec{x}_j, \vec{w}_i \rangle = 0$. Then there exists $c \in \mathbb{R}$ such that $\{\vec{x}_i + c\vec{w}_i\}$ is a vector representation of G.*

Corollary 1.2. *If a graph G is the superposition of the graphs G_1 and G_2, then* $\text{msr}(G) \leq \text{msr}(G_1) + \text{msr}(G_2)$.

1.3. **Matrix Congruence.** If A is a psd matrix and C is full rank, then $A' = C^*AC$ is said to be *congruent* to A and is also psd with the same rank as A. Note that permutation similarity and symmetric row and column operations can be performed by congruence.

Matrix congruence and vector representations are the two main approaches used to obtain our results. Although the two approaches are equivalent, since performing a congruence C^*Y^*YC on a Gram matrix $A = Y^*Y$ can be viewed as operating on the vector representation Y via C, we have found both to possess distinct advantages.

1.4. **Outline.** In Section 2, we present a construction of a positive semidefinite matrix whose graph is a complete bipartite graph, and thereby compute its minimum semidefinite rank. Section 3 uses vector representations and matrix congruence operations to present new results on the effect on minimum semidefinite rank of adding or deleting vertices. In Section 4, we study the effect on minimum semidefinite rank of addition or deletion of an edge. Graphs whose minimum semidefinite rank is at most three are the subject of Section 5. In Section 6, we explore how to construct a positive semidefinite matrix with given graph from a smaller positive semidefinite matrix using related zero/nonzero patterns.

2. Complete Bipartite Graphs

In this section, we use the Schur complement of a matrix to prove that if $K_{m,n}$ is the complete bipartite graph on m and n vertices, then $\text{msr}(K_{m,n}) = \max\{m, n\}$. If E is a block matrix

$$E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A is non-singular and D is square, the matrix $D - CA^{-1}B$ is called the *Schur complement* of A. If E is Hermitian (so that $C = B^*$) and A and $D - B^*A^{-1}B$ are both psd , then E is also psd. Further, the rank of E is given by the sum of the ranks of A and $D - B^*A^{-1}B$ [10].

In what follows, we will often use a $*$-diagram to describe the zero/nonzero pattern of a psd matrix. In these diagrams, a "$*$" represents a nonzero entry (or block of entries), a "$\times$" on the main diagonal represents an arbitrary nonnegative real number, which must be strictly positive if and only if the row in which it appears contains a nonzero entry, and 0 and "?" have their obvious meanings.

Theorem 2.1. *Let $K_{m,n}$ be the complete bipartite graph on m and n vertices. Then* $\text{msr}(K_{m,n}) = \max\{m, n\}$.

Proof. Without loss of generality, we may assume that $m \geq n$. Since $K_{m,n}$ is connected and has an independent set of size m, $\operatorname{msr}(K_{m,n}) \geq m$. To show the reverse inequality, we will construct a psd matrix $B \in \mathcal{P}(K_{m,n})$ having rank m. Consider first an arbitrary psd matrix $A \in \mathcal{P}(K_{m,n})$ where the vertices are ordered so that the independent set of size m comes first. Then A must have the form

$$\left[\begin{array}{c|c} D_1 & * \\ \hline * & D_2 \end{array}\right],$$

where D_1 and D_2 are diagonal matrices that may or may not be the same size.

To construct such a matrix, since the upper-right submatrix is m by n, we choose the n columns to be any n columns from a unitary matrix of size m by m with no zero entries, whose existence is noted, for example, by Severini [16], and call the resulting matrix of column vectors C. If we choose the upper-left and lower-right submatrices to be the correct size identity matrices, our matrix, call it B, looks like

$$B = \left[\begin{array}{c|c} I_m & C \\ \hline C^* & I_n \end{array}\right].$$

By construction, $B \in \mathcal{P}(K_{m,n})$, so that $\operatorname{msr}(K_{m,n}) = m$ as desired. □

3. Vertex Addition and Removal

Recall that a vertex v of a connected graph G is said to be a *cut vertex* of G if the subgraph of G induced by the removal of v is not connected. It is reasonable to think we might manufacture a psd matrix with graph G from psd matrices for the connected components of $G - v$. To accomplish this, we will need two preliminary results. Since the first result is crucial to understanding the second, we include its proof for completeness.

Lemma 3.1 **([13]).** *Let $A = [a_{ij}]$ be a psd matrix, $\alpha \subset \{1, 2, \ldots, n\}$, and $j \notin \alpha$. Then $A[\alpha, \{j\}]^T \in \operatorname{col}(A[\alpha])$.*

Proof. Let A be a psd matrix in $M_n(\mathbb{C})$. We give the proof for $\alpha = \{1, 2, \ldots, n-1\}$. Write

$$A = \left[\begin{array}{c|c} B & \vec{c} \\ \hline \vec{c}^* & d \end{array}\right],$$

where $B = A[\alpha] \in M_{n-1}(\mathbb{C})$. If $\vec{c} \notin \operatorname{col}(B)$, then let $\vec{y}$ be the orthogonal projection of $\vec{c}$ onto $\operatorname{null}(B)$. The vector $\vec{y}$ is nonzero since B is psd, and therefore $\operatorname{null}(B) = \operatorname{col}(B)^{\perp}$. Further, $-\vec{y}^*\vec{c} = -\|\vec{y}\|^2 < 0$. Let $\vec{x} \in \operatorname{null}(B)$ such that $\vec{x}^*\vec{c} < 0$. Then, for sufficiently small $\epsilon > 0$,

$$\begin{bmatrix} \vec{x}^* & \epsilon \end{bmatrix} A \begin{bmatrix} \vec{x} \\ \epsilon \end{bmatrix} < 0$$

which is a contradiction. The more general statement follows. ☐

Lemma 3.2. *If $A \in \mathcal{P}(G)$ and $\operatorname{rank}(A) = \operatorname{msr}(G)$, then every row (column) in A is dependent on the other rows (columns).*

Proof. We need only show that the last row of such a matrix A is dependent on the other rows. The matrix A has the form

$$A = \left[\begin{array}{c|c} A_1 & X \\ \hline X^* & y \end{array}\right]$$

where X is the last column of A without y. By Lemma 3.1, X^* is dependent on the rows of A_1, and so there exist constants c_i such that

$$X^* = c_1 a_1 + \cdots + c_{n-1} a_{n-1}$$

where $a_1, \ldots, a_{n-1}$ are the rows of A_1.

Define a matrix B by the congruence

$$B = C^* \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & 0 \end{array}\right] C = \left[\begin{array}{c|c} A_1 & X \\ \hline X^* & z \end{array}\right]$$

where

$$C = \left[\begin{array}{ccc|c} & & & c_1 \\ & I & & \vdots \\ & & & c_{n-1} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right],$$

I is the identity matrix, and z is some positive number.

Note that $\operatorname{rank}(B) = \operatorname{rank}(A_1)$, $B \in \mathcal{P}(G)$, and the last row of B is dependent on the other rows. If y is such that the last row of A is not dependent on the other rows of A, then $\operatorname{rank}(A) = \operatorname{rank}(B) + 1$ which contradicts our assumption that $\operatorname{rank}(A) = \operatorname{msr}(G)$. Therefore, the last row of A must be dependent on the first $n-1$ rows of A. ☐

Definition 3.3. If G is a graph with a cut vertex v, such that H_1 and H_2 are the connected components of $G - v$, then we write $G = G_1 \centerdot G_2$, where G_1 and G_2 are the subgraphs of G induced by $V(H_1) \cup \{v\}$ and $V(H_2) \cup \{v\}$ respectively.

Theorem 3.4. *If $G = G_1 \centerdot G_2$, then $\operatorname{msr}(G) = \operatorname{msr}(G_1) + \operatorname{msr}(G_2)$.*

Proof. That $\operatorname{msr}(G) \leq \operatorname{msr}(G_1) + \operatorname{msr}(G_2)$ follows from Corollary 1.2, and thus we need only show the reverse inequality.

Let $A \in \mathcal{P}(G)$ have rank equal to $\operatorname{msr}(G)$. The matrix A must have the form

$$A = \left[\begin{array}{c|c|c} A_1 & X & 0 \\ \hline X^* & \times & Z^* \\ \hline 0 & Z & A_2 \end{array}\right]$$

where the rows of A_1 and X^* correspond to the vertices in G_1 and the rows of A_2 and Z^* correspond to the vertices of G_2. Let B_1 be the submatrix of A corresponding to the vertices of G_1 and B_2 the submatrix corresponding to the vertices of G_2.

Suppose that B_2 has minimum rank. Then, by Lemma 3.2, the top row and left column of B_2 are linear combinations of the other rows and columns of B_2 respectively. Hence, we perform a congruence on A, adding this linear combination of the rows columns of B_2 to get

$$\tilde{A} = \left[\begin{array}{c|c|c} A_1 & X & 0 \\ \hline X^* & 0 & 0 \\ \hline 0 & 0 & A_2 \end{array}\right].$$

However, there is zero entry on the diagonal of $\tilde{A}$, but nonzero entries in the same row and column, which is a contradiction since $\tilde{A}$ is psd. Hence, B_2 does not have minimum rank. Similarly, B_1 does not have minimum rank. Hence, $\operatorname{rank}(B_1) \geq \operatorname{msr}(G_1) + 1$ and $\operatorname{rank}(B_2) \geq \operatorname{msr}(G_2) + 1$.

But, since A has minimum rank, the middle row/column of A is dependent on the other rows/columns of A, and so we can perform a congruence on A to eliminate this row and column (while not changing the rank), leaving us with

$$A' = \left[\begin{array}{c|c|c} A_1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & A_2 \end{array}\right].$$

Now, A_1 and A_2 are modified from the matrices B_1 and B_2 by deleting one row and column. Since B_1 and B_2 are psd, deleting a row and corresponding column can decrease the rank by at most 1. Hence,

$$\begin{aligned} \operatorname{rank}(A) = \operatorname{rank}(A_1) + \operatorname{rank}(A_2) &\geq \operatorname{msr}(G_1) + 1 - 1 + \operatorname{msr}(G_2) + 1 - 1 \\ &= \operatorname{msr}(G_1) + \operatorname{msr}(G_2), \end{aligned}$$

and so $\operatorname{msr}(G) \geq \operatorname{msr}(G_1) + \operatorname{msr}(G_2)$, as desired. □

The following corollary has appeared with a different proof [4, Corollary 3.5].

Corollary 3.5. *If a connected graph G has a pendant vertex v, then* $\operatorname{msr}(G) = \operatorname{msr}(G - v) + 1$.

Proof. Let u be the vertex of G that is adjacent to v. If u is adjacent to other vertices as well, then removing u disconnects v from the rest of the graph. Hence, u is a cut vertex, and $G = G - v \,\centerdot\, K_2$, so

$$\operatorname{msr}(G) = \operatorname{msr}(G-v) + \operatorname{msr}(K_2) = \operatorname{msr}(G-v) + 1.$$

Otherwise, G is K_2, and $G - v$ is K_1, and the statement holds. □

Corollary 3.5 is very useful in the context of the overall msr problem, since it means we may restrict our attention to graphs without pendant vertices, as we will see in some of the following results.

Corollary 3.6. *Let G be a graph. If G' is a graph obtained from G by adding a new vertex v and joining v to some of the vertices of G, then* $\operatorname{msr}(G') \leq \operatorname{msr}(G) + d_{G'}(v)$.

Proof. If we view G' as the superposition of G and a star graph G'' with center v, then, by Corollary 1.2, $\operatorname{msr}(G') \leq \operatorname{msr}(G) + d_{G'}(v)$. □

We would like to be able to determine $\operatorname{msr}(G)$ from $\operatorname{msr}(G-v)$ and perhaps some property of v, such as when v is a pendant vertex. This seems difficult in general, but we give a few more results along this line.

Proposition 3.7. *Let G be a connected graph of order $n \geq 4$ with no pendant vertices, and let v be a vertex with $d_G(v) = n-2$. Let w be the unique vertex of G not adjacent to v, and assume that w has no duplicate vertex in $G - v$. Then* $\operatorname{msr}(G-v) = \operatorname{msr}(G)$.

Proof. Because $G - v$ is an induced subgraph of G, $\operatorname{msr}(G-v) \leq \operatorname{msr}(G)$. To show the reverse inequality, given a vector representation of $G - v$, we will construct a vector representation of G having the same rank. Let $\operatorname{msr}(G-v) = m$, let $w_1, \ldots, w_{n-2}$ be the remaining vertices of $G - v$, and let $\{\vec{w}, \vec{w}_1, \ldots, \vec{w}_{n-2}\}$ be a vector representation of $G - v$ in $\mathbb{C}^m$ having rank m. Since v is not adjacent to a pendant vertex, $G - v$ has no isolated vertices, and so this vector representation has no zero vectors. Since no w_j is a duplicate of w, $\vec{w}_j^{\perp} \cap \vec{w}^{\perp} \neq \vec{w}^{\perp}$ for $1 \leq j \leq n-2$, and, since a vector space cannot be written as a finite union of its proper subspaces,

$$\vec{w}^{\perp} \neq \bigcup_{j=1}^{n-2} \left(\vec{w}_j^{\perp} \cap \vec{w}^{\perp} \right) .$$

Therefore, we may choose a nonzero $\vec{v} \in \vec{w}^{\perp}$ such that $\vec{v}$ is not orthogonal to any $\vec{w}_j$. □

Example 3.8. The graph in Figure 1(a) with vertices labeled v and w satisfies the conditions of Proposition 3.7.

Proposition 3.9. *Let G be a connected graph of order $n \geq 5$ with no pendant vertices, and suppose there exists a vertex v such that $d_G(v) = n-3$. Label the vertices of G as $v, v_1, \ldots, v_{n-3}, w_1, w_2,$*

where w_1 and w_2 are not adjacent to v. Moreover, assume for each i and j that w_i, v_j are not duplicate vertices in $G - v$. Finally suppose that one of the following three conditions holds:

(i) *For each i, there exists a j such that v_j is adjacent to exactly one of w_1, w_2, and v_i;*
(ii) *Vertices w_1 and w_2 are not adjacent, and for each i, v_i is not adjacent to both w_1 and w_2;*
(iii) *For each i, the set $\overline{N}(v) \setminus \overline{N}(v_i)$ has cardinality at most $\operatorname{msr}(G-v) - 3$.*

Then $\operatorname{msr}(G) = \operatorname{msr}(G - v)$.

Proof. The vertices w_1 and w_2 are duplicate vertices in G if and only if they are duplicate vertices in $G - v$. In this case, Proposition 3.7 may be applied to $G - w_2$, showing that

$$\operatorname{msr}(G-v) = \operatorname{msr}(G-v-w_2) = \operatorname{msr}(G-w_2-v) = \operatorname{msr}(G-w_2) = \operatorname{msr}(G).$$

So, we may assume that w_1 and w_2 are not duplicate vertices.

As in the proof of Proposition 3.7, given a vector representation of $G - v$, we will construct a vector representation of G having the same rank. Let $\{\vec{v}_1, \ldots, \vec{v}_{n-3}, \vec{w}_1, \vec{w}_2\}$ be a vector representation of $G - v$ in $\mathbb{C}^m$ having rank m. Since w_1 and w_2 are not duplicate vertices, the vectors $\vec{w}_1$ and $\vec{w}_2$ span a plane P.

First, suppose that no $\vec{v}_i$ lies in P. Let $\vec{v}_i'$ be the orthogonal projection of $\vec{v}_i$ onto $P^\perp$. Since each v_i is adjacent to v but w_1 and w_2 are not, $\vec{v}_i' \neq \vec{0}$ for each i. Because the union of the $\vec{v}_i'^\perp \cap P^\perp$ cannot cover $P^\perp$, there exists a vector $\vec{v} \in P^\perp$ such that $\vec{v}$ is not orthogonal to each $\vec{v}_i'$, and thus, not orthogonal to each $\vec{v}_i$. Thus, it remains to show that each condition implies the existence of a vector representation of $G - v$ so that no $\vec{v}_i$ lies in P.

Let the first condition hold. That is, for each i, there exists a j such that v_j is adjacent to exactly one of w_1, w_2, and v_i. Suppose $\vec{w}_1$, $\vec{w}_2$, and $\vec{v}_i$ all belong to P for some i. If v_j is not adjacent to two of w_1 , w_2, and v_i, then $\vec{v}_j \in P^\perp$ since v_j cannot be a duplicate of either w_1 or w_2. But this implies that v_j is not adjacent to all three vertices, contradicting the assumption.

If the second condition is true, then vertices w_1 and w_2 are not adjacent, and for each i, v_i is not adjacent to both w_1 and w_2. For fixed i, if v_i is adjacent to neither w_1 nor w_2, then $\vec{v}_i \in P^\perp$ and we are done. Therefore, assume without loss of generality, that v_i is adjacent to w_1 and not adjacent to w_2. If $\vec{v}_i \in P$, then $\vec{v}_i$ can be written as a non-trivial linear combination of $\vec{w}_1$ and $\vec{w}_2$. However, $\vec{v}_i$ is orthogonal to $\vec{w}_2$, and hence must be a nonzero multiple of $\vec{w}_1$. This is a contradiction, as v_i and w_1 are not duplicate vertices in $G - v$.

In the third case, for each i, the set $S_i = \overline{N}(v) \setminus \overline{N}(v_i)$ has cardinality at most $\operatorname{msr}(G - v) - 3$. Suppose that, for some i, $\vec{v}_i \in P$. Define

$$I = \bigcap_{v_j \in S_i} \vec{v}_j^{\perp}.$$

Note that the dimension of I is at least $m - |S_i| \geq \operatorname{msr}(G-v) - (\operatorname{msr}(G-v) - 3) = 3$. Since the dimension of P is two, there exists a nonzero vector $\vec{w} \in I \cap P^\perp$, and we may

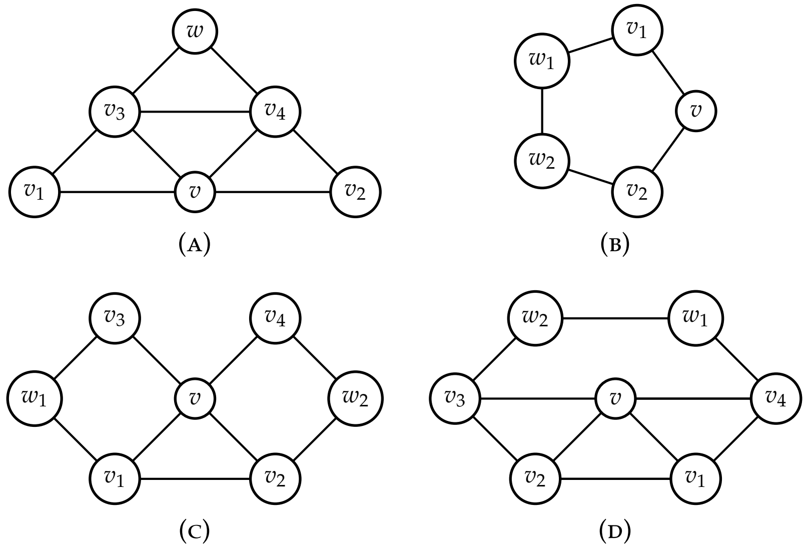

FIGURE 1

add a nonzero constant multiple of $\vec{w}$ to $\vec{v}_i$ to yield a vector $\vec{v}_i'' \in I$ such that $\vec{v}_i'' \notin P$, and which represents v_i in $G - v$. Repeating this process as needed produces a vector representation of $G - v$ with no $\vec{v}_i$ in P. $\square$

Example 3.10. The graphs in Figure 1(b), Figure 1(c), and Figure 1(d), each with a vertex labeled v, satisfy the conditions (i), (ii), and (iii), respectively, of Proposition 3.9.

Proposition 3.11. *Let G be a connected graph, and let v be a vertex of G that has no pendant neighbors. If for each $w \in N(v)$ the number of vertices of G not adjacent to both v and w is less than* $\operatorname{msr}(G - v)$*, then* $\operatorname{msr}(G - v) = \operatorname{msr}(G)$.

Proof. Let $|G - v| = n$. Suppose there exists a vector representation $\vec{V} = \{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq \mathbb{C}^m$ of $G - v$ such that $\vec{v}_i$ can be written as a linear combination of vectors representing vertices in $D(v)$ if and only if v_i is not adjacent to v. Then letting $\vec{D}$ be the span of vectors representing vertices in $D(v)$, and orthogonally projecting each vector of $\vec{V}$ onto $\vec{D}^{\perp}$ yields a vector representation $\vec{V}'$ of $N(v)$. Choosing a vector $\vec{v}$ in $\vec{D}^{\perp}$ that is not orthogonal to each vector of $\vec{V}'$ yields a vector representation $\vec{V} \cup \{\vec{v}\}$ of G of rank m.

If not, choose a vector representation $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq \mathbb{C}^m$ of $G - v$, and let v_i be a neighbor of v in G such that $\vec{v}_i$ can be written as a linear combination of vectors representing vertices in $D(v)$. Let S be the set of vertices adjacent to v but not adjacent to v_i, and $\vec{S}$ the span of vectors representing vertices in S. The cardinality of $S \cup D(v)$ is less than m by the hypothesis. Therefore, the dimension of $\vec{S} + \vec{D}$ is less then m, and there exists a nonzero vector $\vec{w} \in (\vec{S} + \vec{D})^{\perp}$. For a suitably chosen $c \in \mathbb{C}$, $\{\vec{v}_1, \ldots, \vec{v}_i + c\vec{w}, \ldots, \vec{v}_n\}$ is a vector representation of $G - v$, where $\vec{v}_i + c\vec{w} \notin \vec{D}$. Repeating this process yields a vector representation $\vec{V} = \{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq \mathbb{C}^m$ of $G - v$ such that $\vec{v}_i$ can be written as a linear combination of vectors representing vertices in $D(v)$ if and only if v_i is not adjacent to v. □

4. Edge Removal

In this section, we explore the effect on msr of adding or deleting edges from a graph.

Proposition 4.1. *Let the graph G be modified from the superposition of G_1 and G_2 by the removal of an edge common to both subgraphs. Then* $\operatorname{msr}(G) \leq \operatorname{msr}(G_1) + \operatorname{msr}(G_2)$.

Proof. Without loss of generality, if G has n vertices, assume the edge we wish to remove is an edge between the vertices corresponding to rows and columns n and $n-1$. Let $\tilde{A}_1 \in \mathcal{P}(G_1)$ and $\tilde{A}_2 \in \mathcal{P}(G_2)$ each be of minimum rank. Extend the matrices $\tilde{A}_1$ and $\tilde{A}_2$ to A_1 and A_2 such that they have zeros in the rows and columns corresponding to the vertices in G but not in G_i for $i = 1, 2$ (respectively). Then

$$A_1 = \left[\begin{array}{c|cc} ? & \multicolumn{2}{c}{?} \\ \hline ? & \times & a \\ & \bar{a} & \times \end{array}\right] \qquad \text{and} \qquad A_2 = \left[\begin{array}{c|cc} ? & \multicolumn{2}{c}{?} \\ \hline ? & \times & b \\ & \bar{b} & \times \end{array}\right]$$

where $a, b \neq 0$.

First, we perform a congruence on A_1 to get

$$A_1' = \left[\begin{array}{c|cc} I & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 \\ & 0 & -\bar{b}/\bar{a} \end{array}\right] \left[\begin{array}{c|cc} ? & \multicolumn{2}{c}{?} \\ \hline ? & \times & a \\ & \bar{a} & \times \end{array}\right] \left[\begin{array}{c|cc} I & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 \\ & 0 & -b/a \end{array}\right] = \left[\begin{array}{c|cc} ? & \multicolumn{2}{c}{?} \\ \hline ? & \times & -b \\ & -\bar{b} & \times \end{array}\right]$$

which does not alter the graph of A_1.

Now, there is an entry of A_2 with the largest modulus, say m. Let $c \geq 1$ be a positive number large enough such that for all entries a_{ij} of A_1', $|ca_{ij}| > m \geq |z|$ for all entries z of A_2. We now perform a congruence on A_1' to get

$$A_1'' = \left[\begin{array}{c|cc} cI & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array}\right] \left[\begin{array}{c|cc} ? & \multicolumn{2}{c}{?} \\ \hline ? & \times & -b \\ & -\bar{b} & \times \end{array}\right] \left[\begin{array}{c|cc} cI & \multicolumn{2}{c}{0} \\ \hline 0 & 1 & 0 \\ & 0 & 1 \end{array}\right] = \left[\begin{array}{c|cc} \hat{?} & \multicolumn{2}{c}{\hat{?}} \\ \hline \hat{?} & \times & -b \\ & -\bar{b} & \times \end{array}\right]$$

which again does not alter the graph but makes all non-diagonal entries except for $-b$ and $-\bar{b}$ have modulus larger than any entry of A_2 (the $\hat{?}$ indicates that the elements in those sections have been scaled from those in A_1').

So, we now take the sum

$$B = A_1'' + A_2 = \left[\begin{array}{c|cc} \hat{?} & \multicolumn{2}{c}{\hat{?}} \\ \hline \hat{?} & \times & -b \\ & -\bar{b} & \times \end{array}\right] + \left[\begin{array}{c|cc} ? & \multicolumn{2}{c}{?} \\ \hline ? & \times & b \\ & \bar{b} & \times \end{array}\right]$$

which has the graph G because on the overlap of the entries corresponding to G_1 and G_2 (except for the desired removed edge), no entries of A_1'' canceled because the moduli of all entries in A_1'' are larger than the moduli of the entries of A_2. Therefore,

$$\operatorname{rank}(B) = \operatorname{rank}(A_1'' + A_2) \leq \operatorname{rank}(A_1'') + \operatorname{rank}(A_2) = \operatorname{msr}(G_1) + \operatorname{msr}(G_2).$$

Hence $\operatorname{msr}(G) \leq \operatorname{msr}(G_1) + \operatorname{msr}(G_2)$. □

Theorem 4.2. *Addition or removal of an edge from a graph G can change* $\operatorname{msr}(G)$ *by at most 1.*

Proof. Let u and v be vertices of a graph G' that are not adjacent. If G is the graph obtained from G' by adding an edge between u and v, then $\operatorname{msr}(G) \leq msr(G') + 1$ by Corollary 1.2. On the other hand, to remove an edge and obtain G' from G, we can superimpose the edge over G and cancel the common edge. Hence, $\operatorname{msr}(G') \leq \operatorname{msr}(G) + 1$ by Proposition 4.1. It follows that

$$\operatorname{msr}(G) - 1 \leq \operatorname{msr}(G') \leq \operatorname{msr}(G) + 1$$

and

$$\operatorname{msr}(G') - 1 \leq \operatorname{msr}(G) \leq \operatorname{msr}(G') + 1\,.$$

□

Theorem 4.3. *Let v be a vertex of a graph G with no pendant vertices. Let $V = \{v_1, \cdots, v_k\} \subseteq N(v)$. Define $T = \cup_{i=1}^{k} \overline{N}(v_i)$ and assume that $T \subseteq \overline{N}(v)$. If we alter G to obtain a graph G' by deleting the edges from v to some subset of T, then* $\operatorname{msr}(G) \leq \operatorname{msr}(G')$.

Proof. Let $A' \in \mathcal{P}(G')$ have minimum rank. We will begin with the case $V = \{v_1\}$, and we delete the edge from v to v_1. Then

$$A' = \left[\begin{array}{c|c|c} ? & X & Y \\ \hline X^* & \times & 0 \\ \hline Y^* & 0 & \times \end{array}\right]$$

where the X column corresponds to v_1 and the Y column corresponds to v. Note that the "$\times$" in the X row and column is nonzero since v_1 is not a pendant vertex in G and thus is adjacent to another vertex in G'. To construct a matrix A'' that has graph G, we perform a congruence on A' by adding a sufficiently large positive multiple of the X row and column to the Y row and column so that no cancellation occurs. Thus

$$A'' = \left[\begin{array}{c|c|c} ? & X & Z \\ \hline X^* & \times & * \\ \hline Z^* & * & \times \end{array}\right],$$

and A'' has graph G. Thus $\operatorname{msr}(G) \leq \operatorname{rank}(A'') = \operatorname{rank}(A') = \operatorname{msr}(G')$.

Now assume the result holds for the vertex set $\hat{V} = \{v_1, \cdots, v_{k-1}\}$ and the vertex v, and suppose the vertex set $V = \{v_1, \cdots, v_k\}$ and the vertex v satisfy the hypotheses of the theorem. First, for some j, delete edges from v to a subset of $T = \overline{N}(v_j)$ as in the base case. The resulting graph $\tilde{G}$ satisfies $\operatorname{msr}(G) \leq \operatorname{msr}(\tilde{G})$. Now, apply the induction hypothesis to $\tilde{G}$ and $V \setminus \{v_j\}$. □

Corollary 4.4. *Let G be a connected graph, and let v and u be vertices of G. If we modify G to obtain a graph G' by adding edges from v to all vertices in $\overline{N}(u) \setminus \overline{N}(v)$, then* $\operatorname{msr}(G) \geq \operatorname{msr}(G')$.

Proof. If we consider this as deleting edges from G' and apply Theorem 4.3, we have $V = \{u\}$, and $T = \overline{N}(u) \subseteq \overline{N}(v)$. Thus if we delete the edges from v to all vertices in $\overline{N}(u) \setminus \overline{N}(v)$ to get the graph G, then $\operatorname{msr}(G') \leq \operatorname{msr}(G)$. □

Corollary 4.5. *Let G be a connected graph that has no pendant vertices and has vertices $v, v_1, \cdots, v_k$ such that v is adjacent to exactly the vertices in the set*

$$T = \left(\bigcup_{i=1}^{k} \overline{N}(v_i) \right) \setminus \{v\}.$$

Then $\operatorname{msr}(G) = \operatorname{msr}(G - v)$.

Proof. Since $G - v$ is an induced subgraph of G, $\operatorname{msr}(G) \geq \operatorname{msr}(G - v)$. If we delete all edges from v to the vertices of T, we have deleted all of the edges incident to v in G, and so $d'_G(v) = 0$. Hence $\operatorname{msr}(G - v) = \operatorname{msr}(G') \geq \operatorname{msr}(G)$ by Theorem 4.3. □

Corollary 4.6. *Let G be a graph with no pendant vertices, $|G| = n$, and v a vertex of degree $n - 1$. Then* $\operatorname{msr}(G) = \operatorname{msr}(G - v)$.

Proof. Since $d_G(v) = n - 1$, $\overline{N}(v) = G$, and v is adjacent to the vertices in

$$T = \left(\bigcup_{i=1}^{k} \overline{N}(v_i) \right) \setminus \{v\}.$$

Hence, by Corollary 4.5, $\operatorname{msr}(G) = \operatorname{msr}(G - v)$. □

5. Graphs with Small minimum semidefinite rank

We have already seen that $\operatorname{msr}(G) = 1$ if and only if $G = K_n$ with $n \geq 2$ and $\operatorname{msr}(G) = |G| - 1$ if and only if G is a tree. We begin this section with a new proof of a result found in [3] that gives a complete description of graphs G for which $\operatorname{msr}(G) = 2$.

Definition 5.1. Let G be a graph, and u a vertex of G. If u is adjacent to every vertex of $G - u$, we say that G is the *join* of $G - u$ and u, written $(G - u) \vee u$.

Proposition 5.2 ([3]). *Let G be a connected graph without duplicate vertices. Then* $\operatorname{msr}(G) = 2$ *if and only if $G = H_m$ or $H_m \vee v$, where H_m is an $(m - 2)$-regular graph on m vertices.*

Proof. ($\Leftarrow$) Since m must be even, let $V(H_m) = \{v_1, ..., v_{\frac{m}{2}}, u_1, \ldots, u_{\frac{m}{2}}\}$, where v_iv_j, $u_iu_j \in E(H_m)$ for all i and j, $i \neq j$, and $v_iu_j \in E(H_m)$ if and only if $i \neq j$. The degree of each vertex is $m/2 - 1 + m/2 - 1 = m - 2$. Let $\vec{v}_i = (i)\vec{e}_1 + \vec{e}_2$ and $\vec{u}_j = \vec{e}_1 - (j)\vec{e}_2$, where $\vec{e}_1$ and $\vec{e}_2$ are the standard basis vectors in $\mathbb{C}^2$. Then $\langle \vec{v}_i, \vec{u}_j \rangle = i - j = 0$ if and only if $i = j$ as desired. Hence, $\text{msr}(H_m) \leq 2$. Since H_m is not a complete graph, $\text{msr}(H_m) = 2$. Moreover, $H_m \vee u$ and H_m have the same msr by Corollary 4.6.

($\Rightarrow$) The msr of G is at most two if and only if there exists a vector representation of G in $\mathbb{C}^2$. Given a nonzero vector $\vec{z} \in \mathbb{C}^2$, there exists a nonzero vector $\vec{w} \in \mathbb{C}^2$ such that $\langle \vec{z}, \vec{w} \rangle = 0$, and $\vec{w}$ is unique up to a scalar multiple. Hence, if v_i, v_j, $v_k \in V(G)$ and $v_iv_j, v_iv_k \notin E(G)$, then for the vectors $\vec{v}_i$, $\vec{v}_j$, and $\vec{v}_k$ corresponding to v_i, v_j, and v_k, we have $\langle \vec{v}_i, \vec{v}_j \rangle = \langle \vec{v}_i, \vec{v}_k \rangle = 0$ and so $\vec{v}_j = \alpha \vec{v}_k$ for some nonzero α. But then $\langle \vec{v}_j, \vec{v} \rangle = 0$ if and only if $\langle \vec{v}_k, \vec{v} \rangle = 0$ for every vertex v, and so v_j and v_k must be duplicate vertices in G, a contradiction to the assumption. Hence there does not exist such a triple of vertices, and so every vertex of G is not adjacent to at most one other vertex of G. Now write $V(G) = \{u_1, ..., u_m, v_1, ..., v_n\}$, where $d_G(u_i) = m + n - 2$ and $d_G(v_j) = m + n - 1$ for all i and j. If H_m is the subgraph induced by $\{u_1, \ldots, u_m\}$ then H_m is $(m-2)$-regular. There is at most one v_j since all the v_j must be duplicate vertices. Thus, $G = H_m \vee v$ or $G = H_m$. □

Corollary 5.3. *Let G be a connected graph of order $n \geq 4$ which is not complete. If the degree of every vertex of G is at least $n - 2$, then* $\text{msr}(G) = 2$.

Proof. Begin by removing all duplicate vertices to yield a connected graph G' with $\text{msr}(G') = \text{msr}(G)$. If G' is complete, so was G. Further, every vertex of G' has degree at least $|G'| - 2$, and G' can contain only one vertex of degree $|G'| - 1$. By Proposition 5.2, $\text{msr}(G') = 2$. □

The next propositions give partial results concerning graphs with msr three.

Definition 5.4. Let G be a graph that contains $H = K_3$ as an induced subgraph with two of the three vertices having degree two. Then H is said to be a *pendant triangle* of G.

Lemma 5.5. *Let G be a graph of order at least four, not necessarily connected, and is not a complete graph. If the cycle C_m is not a subgraph of $\overline{G}$ for each $m \geq 4$, then $\overline{G}$ has either a pendant vertex or a pendant triangle and G contains at most one isolated vertex. If G has an isolated vertex, every other vertex of G has order at least $|G| - 3$.*

Proof. Let G be a graph of order at least four that is not a complete graph, and assume that $\overline{G}$ does not contain the cycle C_m as a subgraph for all $m \geq 4$. If G contains two isolated vertices u and v, then the subgraph of $\overline{G}$ induced by u and v along with any other two vertices contains a C_4. If G contains an isolated vertex u and a vertex v of degree less than $|G| - 3$, then the subgraph of $\overline{G}$ induced by u, v, and two other neighbors of v in $\overline{G}$ contains a C_4.

To show that $\overline{G}$ has either a pendant vertex or a pendant triangle, we give a proof using induction on the number of three-cycles in $\overline{G}$. The graph $\overline{G}$ contains at least one edge, since we assumed that G is not a complete graph, If $\overline{G}$ has no three-cycles, then $\overline{G}$ is a forest, and therefore contains a pendant vertex. Now, assume the result holds for any G satisfying the hypotheses and $\overline{G}$ has fewer than k three-cycles. Let G be a graph that satisfies the hypotheses and $\overline{G}$ has k three-cycles. From the assumptions, $\overline{G}$ is a chordal graph and must contains a simplicial vertex v. Because $\overline{G}$ cannot contain C_4 as a subgraph, v has at most two neighbors in $\overline{G}$. If v is a pendant vertex or part of a pendant triangle, we are done. Otherwise, v must have two neighbors u and w in $\overline{G}$, both of which have neighbors outside of the maximal clique of $\overline{G}$ containing v. If u and w share a neighbor outside that clique, there is a copy of C_4 in $\overline{G}$, contradicting our assumptions. Thus neither u nor w is a simplicial vertex in $\overline{G} - v$. This means that $\overline{G} - v$ is not a complete graph. Further, as an induced subgraph of $\overline{G}$, $\overline{G} - v$ cannot contain a C_m for any $m \geq 4$. Since $\overline{G} - v$ has $k - 1$ three-cycles, by the induction hypothesis it has either a pendant vertex or a pendant triangle that cannot contain either u or w, and so is also pendant in $\overline{G}$. □

Proposition 5.6. *Let G be a graph. If the cycle C_m is not a subgraph of $\overline{G}$ for each $m \geq 4$, then* $\operatorname{msr}(G) \leq 3$.

Proof. We verify the result by giving an induction argument on n, the order of G. If $n \leq 4$, then $\operatorname{msr}(G) \leq 3$.

Assume that the assertion is true for all graphs of order less than some $n > 4$ that satisfy the hypothesis, and consider a graph G of order n such that the cycle C_m is not a subgraph of $\overline{G}$ for each $m \geq 4$. If G contains an isolated vertex v, $\operatorname{msr}(G) = \operatorname{msr}(G - v) \leq 3$ by the induction hypothesis. Therefore, we may assume that G has no isolated vertices. If G is a complete graph, then $\operatorname{msr}(G) = 1$. Otherwise, by Lemma 5.5, $\overline{G}$ has either a pendant vertex, a pendant triangle, or both.

Suppose first that $\overline{G}$ has a pendant triangle consisting of vertices w_1, w_2, and v, with $d_G(w_1) = d_G(w_2) = 2$. By the induction hypothesis, $\operatorname{msr}(G - \{w_1, w_2\}) \leq 3$. If $G' = G - \{w_1, w_2\}$ contains an isolated vertex u, by Lemma 5.5 it is either the only isolated vertex or G' consists of three isolated vertices. In the latter case, $|G| = 5$ and $\operatorname{msr}(G) \leq 3$. If u is the only isolated vertex, then the degree of every other vertex in G' is at least $n - 5 = |G'| - 3$. Thus the degree of every vertex in $G' - u$ is at least $|G' - u| - 2$, and it follows from Corollary 5.3 that $\operatorname{msr}(G' - u) \leq 2$. Therefore, we may assume that there exists a vector representation $\vec{V}$ of G' in $\mathbb{C}^3$ containing no zero vector. Label the vertices of G as $w_1, w_2, v, v_1, \ldots, v_{n-3}$.

Assume that there exists a vertex of G' other than v that is represented in $\vec{V}$ by a vector parallel to $\vec{v}$. Then all such vertices and v are duplicates in G'. If there exist at least two vertices in G' that are not adjacent to v, then the subgraph of $\overline{G'}$ induced by these two vertices, v, and a duplicate of v contains a C_4. If there is a single vertex v_j of G' that is not adjacent to v, with $\vec{v}_j$ the vector representing v_j in $\vec{V}$, define $\vec{w} = \vec{v}_j$. Otherwise, set $\vec{w} = 0$. Choose a nonzero vector $\vec{u}$ in $\vec{w}^{\perp} \cap \vec{v}^{\perp} \subseteq \mathbb{C}^3$, and consider the vector representation $\vec{V}'$ derived from $\vec{V}$ by adding a nonzero multiple $c\vec{u}$ to each vector representing a duplicate of v, but not to $\vec{v}$. There are only finitely many values of c for which the graph of $\vec{V}'$ is not G'. Fix a value of c for which the graph of $\vec{V}'$ is not G' and denote by $\vec{V}'$ the corresponding vector representation of G' in $\mathbb{C}^3$.

No vertex other than v is represented in $\vec{V}'$ by a vector parallel to $\vec{v}$. Therefore, the subspaces $M_i = \vec{v}^{\perp} \cap \vec{v}_i^{\perp}$ and $N_i = \vec{v}^{\perp} \ominus M_i$ are one-dimensional and $\vec{v}^{\perp} = M_i \oplus N_i$ for each i. Choose a nonzero vector $\vec{w}_1$ orthogonal to $\vec{v}$ and not belonging to M_i or N_i for each i. Then any nonzero $\vec{w}_2$ in $\vec{v}^{\perp}$ orthogonal to $\vec{w}_1$ does not belong to M_i or N_i for each i. Fix any such nonzero $\vec{w}_1$ and $\vec{w}_2$ to yield, along with $\vec{V}'$, a vector representation of G of rank at most three.

Finally, if $\overline{G}$ does not have a pendant triangle, it must have a pendant vertex. Then G is an induced subgraph of a graph F obtained from G by adding a degree-two vertex adjacent to the pendant vertex of G and its neighbor. Therefore, $\operatorname{msr}(G) \leq \operatorname{msr}(F) \leq 3$. □

Lemma 5.7. *If $\overline{G} = P_m$ then there exists a vector representation of G in $\mathbb{C}^3$ with no three vectors in the same plane and no two vectors constant multiples of each other.*

Proof. The proof is by induction on n, the number of vertices of G. The statement is easily verified if $n \leq 4$. Assume the assertion is true for all graphs G for which $\overline{G} = P_n$ and $n < m$, and let G be a graph for which $\overline{G} = P_m$. Label the vertices of G as $v_1, \ldots, v_m$ so that v_1 is a pendant vertex of $\overline{G}$ and v_2 is its neighbor. Using the induction hypothesis, let $\vec{V}$ be a vector representation of $G - v_1$ satisfying the desired conditions.

Since no two of the vectors in $\vec{V}$ are scalar multiples of each other, we may define one-dimensional subspaces $m_{ij} = \vec{v}_2^{\perp} \cap \operatorname{Span}\{\vec{v}_i, \vec{v}_j\}$ for $3 \leq i, j \leq m$. When $i \neq j$, m_{ij} is one-dimensional since $\vec{v}_2$ cannot be orthogonal to both $\vec{v}_i$ and $\vec{v}_j$. The subspaces m_{ij} do not cover $\vec{v}_2^{\perp}$, and so we may choose a nonzero $\vec{v}_1 \in \vec{v}_2^{\perp} \setminus \cup_{i,j} m_{ij}$. Then $\vec{v}_1$ along with $\vec{V}$ gives a vector representation of G in $\mathbb{C}^3$. Furthermore, since $\vec{v}_1$ is not in any m_{ij}, no three vectors in $\vec{V}$ are in the same plane and no two vectors in $\vec{V}$ are scalar multiples of each other. □

Proposition 5.8. *If $\overline{G} = P_n$ with $n \geq 4$, then* $\operatorname{msr}(G) = 3$.

Proof. From Lemma 5.7, $\operatorname{msr}(G) \leq 3$. Since G is not a complete graph, and since the complements of H_m and $H_m \vee u$ are not equal to P_n, $\operatorname{msr}(G) = 3$ by Proposition 5.2. □

Proposition 5.9. *If $\overline{G} = C_n$ with $n \geq 5$ then* $\operatorname{msr}(G) = 3$.

Proof. If $\overline{G} = C_n$, consider $\overline{G} - v = P_{n-1}$ for any vertex v of G. Using Lemma 5.7, select a vector representation $\vec{V}'$ of $G - v$ in $\mathbb{C}^3$ with no three vectors in the same plane and no two vectors constant multiples of each other. Label the vertices of $G - v$ as $v_1, \ldots, v_{n-1}$ so that v_1 and v_{n-1} are adjacent to v in $\overline{G}$. Choose a nonzero vector $\vec{v}$ in $\vec{v}_1^{\perp} \cap \vec{v}_{n-1}^{\perp}$. Recall that no three vectors in $\vec{V}'$ are coplanar, so if there exists a j with $1 < j < n-1$ such that $\vec{v}$ is orthogonal to $\mathrm{Span}\{\vec{v}_1, \vec{v}_j, \vec{v}_{n-1}\}$, then $\vec{v} = \vec{0}$, a contradiction. Therefore, $\vec{v}$ along with $\vec{V}'$ gives a vector representation of G of rank at most three. Since $G - v$ is an induced subgraph of G and $\mathrm{msr}(G - v) = 3$ by Proposition 5.8, we have $\mathrm{msr}(G) = 3$ as desired. □

Proposition 5.10. *Assume that G is a connected graph without duplicate vertices. If* $\mathrm{msr}(G) \leq 3$, *then* $\mathrm{msr}(G) = \mathrm{ts}(G) - 1$.

Proof. The proposition is clear if $\mathrm{msr}(G)$ is one or two. Suppose $\mathrm{msr}(G) = 3$. We may find a vertex v such that there exist vertices $u_1, u_2 \in D(v)$. If $u_1u_2 \notin E(G)$, consider a shortest path from v to u_1 and a shortest path from v to u_2. If the length of either of these paths is three, we are done. If not, we have the paths vw_1u_1 and vw_2u_2. If $w_1 = w_2$, then the vertices v, w_1, u_1, and u_2 induce the desired tree. If w_1 is adjacent to w_2, then the vertices w_1, w_2, u_1, and u_2 induce the desired tree. We cannot have $w_1 \neq w_2$ and $w_1w_2 \notin E(G)$, for then the vertices v, w_1, w_2, u_1, and u_2 induce a tree which would imply that $\mathrm{msr}(G) \geq 4$.

If there do not exist $u_1, u_2 \notin D(v)$ such that $u_1u_2 \notin E(G)$, choose u_1' and u_2' in $D(v)$ such that $u_1'u_2' \in E(G)$. Since u_1' and u_2' are not duplicate vertices, there exists w such that $wu_1' \in E(G)$ and $wu_2' \notin E(G)$. If $w \notin N(v)$, then $D(v)$ contains two vertices which are not adjacent, which would contradict the assumption. Thus $w \in N(v)$ and the vertices v, w, u_1', and u_2' induce a tree with tree size of four. Therefore, $\mathrm{msr}(G) = \mathrm{ts}(G) - 1$. □

6. Further Vertex Removal

Given a connected graph G and a vertex v of G, define the set $S(G, v)$ to be those graphs associated with G and v as $H \in S(G, v)$ if and only if H can be obtained from $G - v$ by modifying the edges between the vertices of $N(v)$ in the following way: if $u, w \in N(v)$ in G and uw is not an edge in $G - v$, then uw is an edge in H, while if uw is an edge in $G - v$ then uw may or may not be an edge in H. This freedom of choice determines $S(G, v)$. Set $s(G, v) = \min\{\mathrm{msr}(H) \mid H \in S(G, v)\}$.

Lemma 6.1. *Let G be a connected graph and let v be a vertex of G. Then,* $\mathrm{msr}(G) \geq s(G, v) + 1$.

Proof. Let G have order n and assume the n-by-n matrix $A \in \mathcal{P}(G)$ has minimum rank. Assume that $V(G)$ has been ordered so that v is listed last and the vertices in $N(v)$ appear immediately before v. Then $A = [a_{ij}]$ has the form

$$A = \left[\begin{array}{c|ccc|c} ? & & ? & & 0 \\ \hline & \times & & ? & \\ ? & & \ddots & & * \\ & ? & & \times & \\ \hline 0 & & * & & \times \end{array}\right].$$

We will perform a series of congruences on A. First, we add a multiple of the last row (column) to the next row (column) to make $a_{n-1,n} = a_{n,n-1} = 0$. Continue with this process until all of the stars in the last row and column of A have been changed to zeros. This gives the new matrix

$$A' = \left[\begin{array}{c|ccc|c} ? & & ? & & 0 \\ \hline & \times & & ? & \\ ? & & \ddots & & 0 \\ & ? & & \times & \\ \hline 0 & & 0 & & \times \end{array}\right].$$

During the congruence operation on the ith row and column, we add a nonzero number to all zero entries in the ith row and column of A, and the nonzero entries in the ith row and column may or may not have been changed to zeros. Hence, the submatrix $A'(n)$ obtained from A' by deleting the last row and column of A' has graph $H \in S(G, v)$. Since A' is a direct sum of $A'(n)$ and a nonzero one-by-one matrix, $\operatorname{rank}(A') = \operatorname{rank}(A'(n)) + 1$. Hence

$$\operatorname{msr}(G) = \operatorname{rank}(A) = \operatorname{rank}(A') = \operatorname{rank}(A'(n)) + 1 \geq \operatorname{msr}(H) + 1 \geq s(G, v) + 1\,.$$

□

Example 6.2. Consider the complete bipartite graph $K_{2,n}$. For $n \geq 3$, $\operatorname{msr}(K_{2,n}) = n$ by Theorem 2.1. If v belongs to the smaller independent set of vertices, then $S(G, v)$ consists of only K_{n+1}, and $s(G, v) = 1$. Therefore, $\operatorname{msr}(G) - s(G, v)$ could be arbitrarily large.

Theorem 6.3. *Let G be a connected graph. Assume that, for some vertex v of G, the subgraph of G induced by $N(v)$ is either a clique or is modified from a clique by deleting either one or two edges. Then* $\operatorname{msr}(G) = s(G, v) + 1$.

Proof. Since $\operatorname{msr}(G) \geq s(G, v) + 1$ by Lemma 6.1, we need only show that $\operatorname{msr}(G) \leq s(G, v) + 1$.

We will separate the proof into four cases.

Case 1: The subgraph of G induced by $N(v)$ is a clique. If we superimpose the clique induced by $N(v)$ upon any $H \in S(G, v)$ the resultant graph is G. Since a clique has msr 1, $\operatorname{msr}(G) \leq \operatorname{msr}(H) + 1$ for all $H \in S(G, v)$ by Corollary 1.2. Thus $\operatorname{msr}(G) \leq s(G, v) + 1$.

Case 2: The graph induced by $N(v)$ is modified from a clique by removing one edge. Order the vertices of $V(G)$ so that the vertex v is last, the endpoints of the missing edge are immediately before v, and the remaining vertices in $N(v)$ immediately precede those two. Assume $A \in \mathcal{P}(H)$ where $H \in S(G, v)$ and $\operatorname{msr}(H) = s(G, v)$. Now extend A by a row and column with a nonzero entry on the diagonal and zeros elsewhere to construct a matrix A_J of the form

$$A_J = \left[\begin{array}{c|c|c|c|c} ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & * & 0 \\ \hline ? & ? & * & \times & 0 \\ \hline 0 & 0 & 0 & 0 & \times \end{array}\right].$$

Note that $\operatorname{rank}(A_J) = \operatorname{rank}(A) + 1$. Next, add the last row (column) to the penultimate row (column) to get

$$A_J' = \left[\begin{array}{c|c|c|c|c} ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & * & 0 \\ \hline ? & ? & * & \times & * \\ \hline 0 & 0 & 0 & * & \times \end{array}\right].$$

Now, add an appropriate multiple of the last row (column) to row (column) $n-2$, to change the $n-2, n-1$ and $n-1, n-2$ entries to zero. The corresponding matrix A_J'' has the form

$$A_J'' = \left[\begin{array}{c|c|c|c|c} ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & 0 & * \\ \hline ? & ? & 0 & \times & * \\ \hline 0 & 0 & * & * & \times \end{array}\right].$$

We then perform multiple congruences, adding multiples of the last row and column to the rest of the rows and columns corresponding to the remaining vertices in the neighborhood of v, each time adding a high enough multiple so as not to cancel any of the nonzero entries. We get

$$A_J''' = \left[\begin{array}{c|c|c|c|c} ? & ? & ? & ? & 0 \\ \hline ? & * & * & * & * \\ \hline ? & * & \times & 0 & * \\ \hline ? & * & 0 & \times & * \\ \hline 0 & * & * & * & \times \end{array}\right].$$

Note that A_J''' has graph G and so

$$\operatorname{msr}(H) + 1 = \operatorname{rank}(A_J) = \operatorname{rank}(A_J''') \geq \operatorname{msr}(G)\,.$$

Since $\operatorname{msr}(H) = s(G,v)$, $\operatorname{msr}(G) \leq s(G,v) + 1$.

Case 3: The graph induced by $N(v)$ is modified from a clique by deleting two edges that have a common vertex. Order the entries of $V(G)$ so that the vertex v is last, the distinct endpoints of the two missing edges are next, preceded by the common vertex and then the remaining vertices in $N(v)$.

We begin with a matrix $A \in \mathcal{P}(H)$ of minimum rank, where $H \in S(G,v)$ and $\operatorname{msr}(H) = s(G,v)$. Consider

$$A_I = \left[\begin{array}{c|c|c|c|c|c} ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & a & b & 0 \\ \hline ? & ? & \overline{a} & \times & c & 0 \\ \hline ? & ? & \overline{b} & \overline{c} & \times & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

where $A = A_I(\{n\})$. Note that $\operatorname{rank}(A_I) = \operatorname{msr}(H) + 1$. Let d be a positive real number such that $|d^2ab| > |c|$. Perform a congruence on A_I to get

$$\tilde{A}_I = \left[\begin{array}{c|c|c|c|c|c} I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & -db \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] A_I \left[\begin{array}{c|c|c|c|c|c} I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & -\overline{db} & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c|c} ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & a & b & 0 \\ \hline ? & ? & \overline{a} & \times & c & 0 \\ \hline ? & ? & \overline{b} & \overline{c} & \times & -db \\ \hline 0 & 0 & 0 & 0 & -\overline{db} & 1 \end{array}\right].$$

We then perform another congruence to get

$$\tilde{\tilde{A}}_I = \left[\begin{array}{c|c|c|c|c|c} I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & -da \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tilde{A}_I \left[\begin{array}{c|c|c|c|c|c} I & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -\overline{da} & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c|c|c} ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & a & b & 0 \\ \hline ? & ? & \overline{a} & \times & c + dad\overline{b} & -da \\ \hline ? & ? & \overline{b} & \overline{c} + d\overline{a}db & \times & -db \\ \hline 0 & 0 & 0 & -\overline{da} & -\overline{db} & 1 \end{array}\right].$$

Note that since $|d^2ab| > |c|$, $c + dad\overline{b} \neq 0$. Now we perform one last congruence, adding a multiple of the last row and column to the fourth to last row and column, to yield the pattern

$$\overline{A} = \left[\begin{array}{c|c|c|c|c|c} ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & 0 & 0 & * \\ \hline ? & ? & 0 & \times & * & * \\ \hline ? & ? & 0 & * & \times & * \\ \hline 0 & 0 & * & * & * & \times \end{array}\right].$$

Finally, we proceed to add sufficiently large multiples of the last row and column to the remaining rows and columns corresponding to vertices in $N(v)$ to get

$$\overline{\overline{A}} = \left[\begin{array}{c|c|c|c|c|c} ? & ? & ? & ? & ? & 0 \\ \hline ? & * & * & * & * & * \\ \hline ? & * & \times & 0 & 0 & * \\ \hline ? & * & 0 & \times & * & * \\ \hline ? & * & 0 & * & \times & * \\ \hline 0 & * & * & * & * & \times \end{array}\right].$$

Since $\overline{\overline{A}} \in \mathcal{P}(G)$,

$$\mathrm{msr}(G) \leq \mathrm{rank}(\overline{\overline{A}}) = \mathrm{rank}(A) + 1 = s(G, v) + 1\,.$$

Case 4: The graph induced by $N(v)$ is modified from a clique by deleting two edges that have distinct end points. Order the vertices of $V(G)$ so that the vertex v is last, the four distinct endpoints of the two missing edges are next, and then the remaining vertices in $N(v)$. As before, we begin with a matrix $A \in \mathcal{P}(H)$ of minimum rank where $H \in S(G, v)$ and $\mathrm{msr}(H) = s(G, v)$. Extend A to a matrix A_I by adding a row and column whose entries are zero except for a one on the diagonal. Then A_I has the form

$$A_I = \left[\begin{array}{c|c|c|c|c|c|c} ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & a & b & c & 0 \\ \hline ? & ? & \bar{a} & \times & d & e & 0 \\ \hline ? & ? & \bar{b} & \bar{d} & \times & f & 0 \\ \hline ? & ? & \bar{c} & \bar{e} & \bar{f} & \times & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right],$$

where a and f are nonzero and all other variables can be any complex number. We add the last row and column to the row and column next to them to get

$$\tilde{A}_I = \left[\begin{array}{c|c|c|c|c|c|c} ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & a & b & c & 0 \\ \hline ? & ? & \overline{a} & \times & d & e & 0 \\ \hline ? & ? & \overline{b} & \overline{d} & \times & f & 0 \\ \hline ? & ? & \overline{c} & \overline{e} & \overline{f} & \times & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right].$$

Next we add $-f$ times the last row to the third to last row and then add $-\overline{f}$ times the resulting last column to the third to last column to get

$$\tilde{\tilde{A}}_I = \left[\begin{array}{c|c|c|c|c|c|c} ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & a & b & c & 0 \\ \hline ? & ? & \overline{a} & \times & d & e & 0 \\ \hline ? & ? & \overline{b} & \overline{d} & \times & 0 & -f \\ \hline ? & ? & \overline{c} & \overline{e} & 0 & \times & 1 \\ \hline 0 & 0 & 0 & 0 & -\overline{f} & 1 & 1 \end{array}\right].$$

Now we choose two numbers x and y such that $\overline{x}y = -a$, with $|x|$ sufficiently large so that $|xf| > |d|$ and $|x| > |e|$, and $|y|$ small enough so that $|yf| < |b|$ and $|y| < |c|$.

We then add x times the last row to the fourth to last row and $\overline{x}$ times the resulting last column to the fourth to last column to get

$$\tilde{\tilde{\tilde{A}}}_I = \left[\begin{array}{c|c|c|c|c|c|c} ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & a & b & c & 0 \\ \hline ? & ? & \overline{a} & \times & d - x\overline{f} & e + x & x \\ \hline ? & ? & \overline{b} & \overline{d} - \overline{x}f & \times & 0 & -f \\ \hline ? & ? & \overline{c} & \overline{e} + \overline{x} & 0 & \times & 1 \\ \hline 0 & 0 & 0 & \overline{x} & -\overline{f} & 1 & 1 \end{array}\right].$$

We add y times the last row to the fifth to last row and then add $\overline{y}$ times the resulting last column to the fifth to last column to get

$$A' = \left[\begin{array}{c|c|c|c|c|c|c} ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & a+\overline{x}y & b-\overline{f}y & c+y & y \\ \hline ? & ? & \overline{a}+x\overline{y} & \times & d-x\overline{f} & e+x & x \\ \hline ? & ? & \overline{b}-\overline{y}f & \overline{d}-\overline{x}f & \times & 0 & -f \\ \hline ? & ? & \overline{c}+\overline{y} & \overline{e}+\overline{x} & 0 & \times & 1 \\ \hline 0 & 0 & \overline{y} & \overline{x} & -\overline{f} & 1 & 1 \end{array}\right].$$

From the definitions of x and y, we see that A' has the pattern

$$A' = \left[\begin{array}{c|c|c|c|c|c|c} ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & ? & \times & 0 & * & * & * \\ \hline ? & ? & 0 & \times & * & * & * \\ \hline ? & ? & * & * & \times & 0 & * \\ \hline ? & ? & * & * & 0 & \times & * \\ \hline 0 & 0 & * & * & * & * & \times \end{array}\right].$$

We now perform a series of congruences as in the previous cases by adding sufficiently large multiples of the last row and column to the remaining rows and columns of $N(v)$ to get

$$A'' = \left[\begin{array}{c|c|c|c|c|c|c} ? & ? & ? & ? & ? & ? & 0 \\ \hline ? & * & * & * & * & * & * \\ \hline ? & * & \times & 0 & * & * & * \\ \hline ? & * & 0 & \times & * & * & * \\ \hline ? & * & * & * & \times & 0 & * \\ \hline ? & * & * & * & 0 & \times & * \\ \hline 0 & * & * & * & * & * & \times \end{array}\right],$$

which has the graph G. So

$$\operatorname{msr}(G) \leq \operatorname{rank}(A'') = \operatorname{rank}(A_I) = s(G, v) + 1\,.$$

Having shown the result to be true in all four cases, the proof is complete. □

Corollary 6.4. *If G is a connected graph with a vertex v such that the subgraph of G induced by $N(v)$ forms a clique, G' is obtained from G by deleting an edge from that clique, and G'' is modified from G' by deleting another edge between two neighbors of v, then* $\operatorname{msr}(G) \leq \operatorname{msr}(G') \leq \operatorname{msr}(G'')$. *Furthermore,* $\operatorname{msr}(G') \leq \operatorname{msr}(G) + 1$ *and* $\operatorname{msr}(G'') \leq \operatorname{msr}(G') + 1$.

Proof. From Theorem 6.3, we know that $\operatorname{msr}(G) = s(G, v) + 1$, $\operatorname{msr}(G') = s(G', v) + 1$, and $\operatorname{msr}(G'') = s(G'', v) + 1$, and $S(G'', v) \subset S(G', v) \subset S(G, v)$. Since we are taking minimums over these sets, $\operatorname{msr}(G) \leq \operatorname{msr}(G') \leq \operatorname{msr}(G'')$. The second part of the corollary follows from Theorem 4.2. □

Note that Theorem 6.3 may always be applied to a vertex of degree two, including the following case:

Corollary 6.5 ([11])**.** *Suppose a graph G has a vertex v of degree two that is adjacent to vertices u and w and suppose furthermore that u and w do not share an edge. Then* $\operatorname{msr}(G) = \operatorname{msr}(G') + 1$, *where G' is modified from G by removing the vertex v and adding an edge from u to w.*

Corollary 6.6 ([4, p. 733], c.f. [17, Theorem 4.3])**.** *For the cycle C_n,* $\operatorname{msr}(C_n) = n - 2$.

Proof. Note that C_3 is the same as K_3 and so has msr 1. Now assume that C_k has msr $k - 2$, and consider the cycle C_{k+1}. Any one of the vertices of C_{k+1} has a neighborhood of two nonadjacent vertices. So, if we focus on a vertex v, by Corollary 6.5 removing v and placing an edge between its neighbors yields C_k. So $\operatorname{msr}(C_{k+1}) = \operatorname{msr}(C_k) + 1 = k - 1$. Hence, by induction, the corollary is true for all integers n. □

We can also find the msr of the wheel with n vertices by noticing that the wheel is modified from C_{n-1} by adding a vertex that is connected to all other vertices (that is, the new vertex becomes a vertex of degree $n - 1$ in a graph with n vertices), which does not change msr by Corollary 4.6. So the wheel with n vertices has msr $n - 3$.

In addition, we can also use the discussion on degree two vertices along with what we know about pendant vertices and vertices of degree 0 to find the msr of all partial 2-trees. Recall that a *k-tree* is a graph that can be constructed from a k-clique by adding one vertex at a time adjacent to exactly the vertices in an already existing k-clique [9,18]. Thus a traditional tree is a 1-tree, and 2-trees have triangles as building blocks and the superposition is along edges. A *partial k-tree* is a k-tree from which some edges (without incident vertices) have been deleted.

If we start with a partial 2-tree G, where G' is a 2-tree with G as a subgraph, we can look at a vertex v in G' that forms a leaf (where we can define a "leaf" as being a vertex in a 2-tree with degree two). It is important to note that if we remove a leaf from a 2-tree, the resulting graph is a 2-tree (unless the 2-tree is C_3). In G, $d_G(v) = 0$, 1, or 2. If $d_G(v) = 0$, $G - v$ is also a partial 2-tree and we know that deleting a vertex of degree 0 does not change msr If $d_G(v) = 1$, we can delete v from G to get another partial 2-tree and decrease the msr by 1. If $d_G(v) = 2$ and v is adjacent to vertices u and w, then we need know at most the msr of $\tilde{G}$ and msr of $\hat{G}$ where $\tilde{G}$ is the graph modified from G where v is removed and u and w share an edge and $\hat{G}$ is the graph modified from G where v is removed and u and w do not share an edge. Both $\tilde{G}$ and $\hat{G}$ are partial linear 2-trees since in G' there is an edge between u and w and v is a leaf. We thus can simplify finding the msr of any partial 2-tree to that of finding the msr of a finite number of partial 2-trees with fewer vertices. We thus can repeat this process until we get to a set of partial 2-trees with 3 vertices, for which we can find the msr fairly easily. Hence, we have a recursive process to find the msr of all partial 2-trees.

If a graph G has a vertex v with $d_G(v) = 3$, and $N(v)$ forms a clique, then, by the theorem, $\operatorname{msr}(G) = s(G, v) + 1 = \operatorname{msr}(G') + 1$ for some $G' \in S(G, v)$. If $d_v(G) = 3$, the induced subgraph of $N(v)$ has 2 edges, then $N(v)$ forms a graph modified from a clique by removing one edge, and $\operatorname{msr}(G) = s(G, v) + 1 = \operatorname{msr}(G'') + 1$ for some $G'' \in S(G, v)$ as before, except G'' is modified from G by replacing the missing third edge in $N(v)$ and perhaps removing one or both of the edges that are already present in the induced subgraph of $N(v)$.

We can similarly simplify if $d_G(v) = 3$ and $N(v)$ has one edge since the induced subgraph of $N(v)$ is modified from a 3-clique by removing two edges. The only case for degree three vertices where Theorem 6.3 does not apply is the case when $N(v)$ is an independent set, as the induced subgraph of $N(v)$ is modified from a clique by removing three edges.

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