On the Relative Position of Multiple Eigenvalues in the Spectrum of an Hermitian Matrix with a Given Graph

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Abstract

For Hermitian matrices, whose graph is a given tree, the relationships among vertex degrees, multiple eigenvalues and the relative position of the underlying eigenvalue in the ordered spectrum are discussed in detail. In the process, certain aspects of special vertices, whose removal results in an increase in multiplicity are investigated.

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1 Introduction

It is known that the graph of a real symmetric matrix can substantially limit the possible multiplicities of the eigenvalues. For example, it is well known that in an irreducible, symmetric tridiagonal matrix (the graph is a path), each eigenvalue has multiplicity 1, and, more generally [JD99], if the graph is a tree, no multiplicity is greater than the "path cover number" (best possible). It is our purpose here to show that, in addition, even when certain multiplicities are possible, the graph can impose restrictions on the numerical order of the eigenvalues attaining these multiplicities. There has been a hint of this phenomenon previously in that it has been observed (e.g. [JD99]) that if the graph is any tree, the largest and smallest eigenvalues must have multiplicity 1. We will show that the restrictions go much deeper than this, and the prior fact will, in a quite new way, be a very special case of our observations.

Recall that the (undirected) graph $G = \mathcal{G}(A)$ of an n-by-n real symmetric (or complex Hermitian) matrix $A = (a_{ij})$ has vertices $1, 2, \ldots, n$ and an edge between i and j, $i \neq j$, if and only if a_{ij} $(= a_{ji}) \neq 0$. In our case, the graph has no loops (self-edges) and is independent of the diagonal of A. In this spirit we consider all the real symmetric matrices with a given graph G and call this set of matrices $\mathcal{S}(G)$; thus, $A \in \mathcal{S}(G)$ if and only if $\mathcal{G}(A) = G$. We shall primarily be concerned with the case in which G is a tree T. In this event, we could, as well, include complex Hermitian matrices in our results (each Hermitian matrix whose graph is a tree is diagonally unitarily similar to a real symmetric matrix, with nonnegative off-diagonal entries and the same graph), but, for simplicity, we shall confine discussions to the real symmetric case.

Generally, we are interested in the very large problem of determining for each graph G, what lists of multiplicities, ordered by the numerical order of the underlying eigenvalues, can occur in $\mathcal{S}(G)$. (We suspect that this is equivalent to the inverse eigenvalue problem for G.) However, here, we shall limit our attention to local statements about the relative position of one or two multiple eigenvalues in an ordered multiplicity list.

We first ask how far to the left an eigenvalue of given multiplicity may occur

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among the eigenvalues for any $A \in \mathcal{S}(T)$, T a given tree. A precise theoretical answer is given, and this theoretical answer is then applied to give some practical statements. It follows that the eigenvalue multiplicities of A constrain the vertex degree sequence of $\mathcal{G}(A)$, if the graph is a tree. We then turn attention to the number of eigenvalues between two eigenvalues of given multiplicity and close by applying these ideas to some special classes of trees, including giving all possible ordered multiplicity lists for trees that are stars. Along the way careful attention must be paid to vertices of degree ≥ 3 in relation to multiple eigenvalues. "Parter vertices" are introduced in the next section.

2 Background

We record here three known results and related ideas that will be important for our results. First are the fundamental interlacing inequalities (e.g. [HJ85]) for the eigenvalues of an Hermitian matrix and a principal submatrix of size one smaller.

Theorem 2.1 *If A is an n-by-n Hermitian matrix with eigenvalues*

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$$

and $A(i)$, the $(n-1)$-by-$(n-1)$ principal submatrix of A resulting from deletion of row and column i, has eigenvalues

$$\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n-1},$$

then

$$\alpha_1 \leq \beta_1 \leq \alpha_2 \leq \beta_2 \leq \cdots \leq \alpha_{n-1} \leq \beta_{n-1} \leq \alpha_n.$$

Thus, the eigenvalues of a principal submatrix are closely related to those of the full matrix. The particular feature of that relationship that interests us most is the following. Let $m_A(\lambda)$ denote the (algebraic) multiplicity of an eigenvalue λ in the spectrum ($\sigma(A)$) of the n-by-n matrix A.

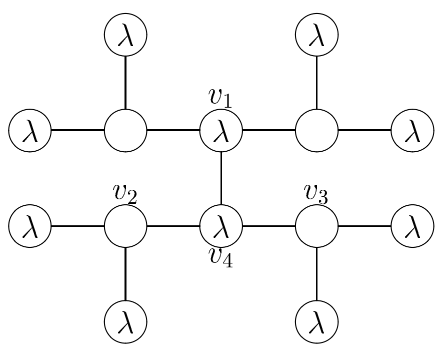
Corollary 2.2 *Let A be an n-by-n Hermitian matrix and suppose that $m_A(\lambda) = m$. If $A(i)$ is the principal submatrix of A resulting from deletion of row and column i, then $m_{A(i)}(\lambda) \in \{m-1, m, m+1\}$.*

In other words, the multiplicity of an eigenvalue can change by at most 1 if a principal submatrix (of 1 smaller dimension) is extracted. A natural guess is that a decrease of 1 in multiplicity is common. However, there is a lovely surprise, at least when the graph of Hermitian A is a tree [Par60,Wie84].

Theorem 2.3 *Suppose that A is an n-by-n Hermitian matrix whose graph is a tree T. If $m_A(\lambda) \geq 2$, then there is a vertex v of T, $\deg(v) \geq 3$, such that $m_{A(v)}(\lambda) = m_A(\lambda) + 1$ and λ is an eigenvalue of at least three of the direct summands of $A(v)$.*

We call any vertex v meeting the requirements of theorem 2.3's conclusion a *Parter vertex* of T for λ relative to A (a Parter vertex, for short). Thus, a Parter vertex has degree at least 3, and it is not difficult, for any degree 3 vertex v in any tree T, to construct an example A with a multiple eigenvalue λ for which v is Parter. If only $m_{A(v)}(\lambda) = m_A(\lambda) + 1$, $m_A(\lambda) \geq 1$, we call the vertex v *weak Parter*. For much of our discussion, this notion is sufficient. It can also happen that after a Parter vertex v is removed from T, there are still (weak) Parter vertices in some of the remaining branches (components of T). In the same spirit, we call a set of vertices $\{v_1, \ldots, v_k\}$ of T a *Parter set* of vertices of T for λ relative to A (a Parter set, for short) if $m_A(\lambda) \geq 1$ and $m_{A(v_1,\ldots,v_k)}(\lambda) = k + m_A(\lambda)$. Here, as throughout, $A(v_1, \ldots, v_k)$ denotes the principal submatrix of A resulting from deletion of the rows and columns corresponding to $\{v_1, \ldots, v_k\}$. It is immediate (because of corollary 2.2) that each vertex in a Parter set of vertices must be individually weak Parter: for the multiplicity to increase by k, by corollary 2.2 it would have to increase by 1 with the removal of each vertex, starting with any one. However, a collection of Parter vertices does not necessarily form a Parter set.

Example 2.4 Suppose a 14-by-14 Hermitian A has graph



and every diagonal entry corresponding to a labelled vertex has value λ.

Corollary 3.8 and the proof of theorem 3.4 will show that there exists a Parter set for λ consisting of three degree 3 vertices. By inspection, removing three degree 3 vertices results in a multiplicity at most 7. In fact, $m_{A(v_1,v_2,v_3)}(\lambda) = 7$, so $m_A(\lambda) = 4$, and v_1 is Parter for λ. By a similar argument, v_4 is also Parter for λ. However, $m_{A(v_1,v_4)}(\lambda) = 4$.

Finally, the maximum multiplicity of any single eigenvalue among all matrices in $\mathcal{S}(T)$, T a tree, is known [JD99]. The path cover number $p(T)$ of a tree T is the minimum number of vertex disjoint paths of T that cover all the vertices of T.

Theorem 2.5 *If T is a tree, the maximum multiplicity occurring for any eigenvalue in any $A \in \mathcal{S}(T)$ is $p(T)$.*

The value of $p(T)$ may be determined by a simple and cheap algorithm.

3 Relative Position of a Single Multiple Eigenvalue

Definition 3.1 Let T be a tree and m a positive integer. Then, $k(m,T)$ is the smallest nonnegative integer k such that there exist k distinct vertices $v_1, \ldots, v_k$ of T whose removal from T leaves at least $m+k$ components. (If there is no such k, call $k(m,T)\ +\infty$.)

Observation 3.2 Suppose that the tree T has n vertices and degree sequence: $d_1 \geq d_2 \geq \cdots \geq d_n$, and let $k'(m,T)$ be the least k such that $1+\sum_{i=1}^{k}(d_i-2) \geq m$. (If there is no such k, call $k'(m,T)\ +\infty$.)

(1) Then, $k(m,T) \geq k'(m,T)$. Equality is attained if T is *segregated* (no two vertices of degree ≥ 3 are adjacent).
(2) In particular, if T is binary (no vertex of degree > 3) and segregated, $k(m,T) = k'(m,T) = m-1$, as long as T has at least $m-1$ degree three vertices. Note that, by theorem 2.5, if T is binary and segregated, the maximum multiplicity of an eigenvalue in $A \in \mathcal{S}(T)$ is one more than the number of degree 3 vertices.

In order to discuss the relative position of an eigenvalue, we regard the spectrum of an Hermitian matrix as an ordered list. Then, we may denote the number of eigenvalues of A strictly to the left (right) of a real number λ by $l_A(\lambda)$ ($r_A(\lambda)$). As $A \in \mathcal{S}(T)$ if and only if $-A \in \mathcal{S}(T)$, statements about $l_A(\lambda)$ are often equivalent to ones about $r_A(\lambda)$. We also denote by $b_A(\lambda_1, \lambda_2)$ the number of eigenvalues of A strictly between λ_1 and λ_2.

Lemma 3.3 *If T is a tree and $A \in \mathcal{S}(T)$ is such that $m_A(\lambda) \geq 1$ and $\{v_1, \ldots, v_k\}$ is a Parter set of vertices of T for $\lambda \in \sigma(A)$, then*

$$l_A(\lambda) \geq k \text{ and } r_A(\lambda) \geq k.$$

PROOF. Since $\{v_1, \ldots, v_k\}$ is a Parter set, $m_{A(\{v_1,\ldots,v_{i+1}\})}(\lambda) = m_{A(\{v_1,\ldots,v_i\})}(\lambda)+1$, $i = 1, \ldots, k-1$, and $m_{A(\{v_1\})}(\lambda) = m_A(\lambda)+1$. By considering the interlacing inequalities, for each i the number of eigenvalues to the left (right) of λ

decreases by 1 each time a vertex is removed. Thus, there must have been at least k initially. $\square$

Theorem 3.4 *Let T be a tree and $A \in \mathcal{S}(T)$. If $m = m_A(\lambda) \geq 1$, then*

$$l_A(\lambda) \geq k(m, T) \text{ and } r_A(\lambda) \geq k(m, T).$$

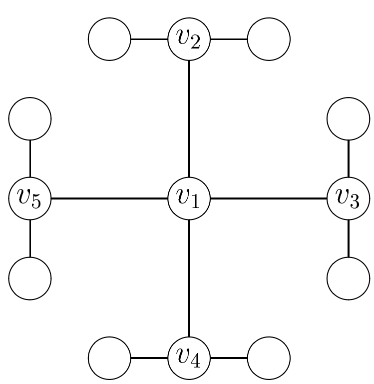
Moreover, if there is a matrix $B \in \mathcal{S}(T)$ with an eigenvalue λ of multiplicity m, then there is a matrix $C \in \mathcal{S}(T)$ with $m_C(\lambda) \geq m$ and $l_C(\lambda) = k(m, T)$; similarly, there is a $C' \in \mathcal{S}(T)$ with $m_{C'}(\lambda) \geq m$ and $r_{C'}(\lambda) = k(m, T)$.

PROOF. The case $m_A(\lambda) = 1$ is trivial, since $k(m, T) = 0$. If $m_A(\lambda) \geq 2$, there exists a Parter vertex v_1 for λ. When the principal submatrix $A(v_1)$ is extracted, λ may be a multiple eigenvalue of one of the direct summands of this matrix. If so, another Parter vertex v_2 may be found. Because $m_{A(v_1,v_2)}(\lambda) = m + 2$, v_1 and v_2 form a Parter set. Continue adding Parter vertices in this manner, obtaining a Parter set $\{v_1, \ldots, v_p\}$ for λ such that the multiplicity of λ in each direct summand of $A(v_1, \ldots, v_p)$ is at most 1.

Because $m_{A(v_1,\ldots,v_p)}(\lambda) = m+p$, and λ is not a multiple eigenvalue of any direct summand of $A(v_1, \ldots, v_p)$, the graph $T - \{v_1, \ldots, v_p\}$ must have at least $m+p$ components. Thus, the removal of p vertices $\{v_1, \ldots, v_p\}$ from T leaves at least $m+p$ components. By definition, $k(m, T)$ is the minimum number of vertices exhibiting this behavior. Therefore, $p \geq k(m, T)$. The lemma completes the proof of the necessary conditions.

In order to construct C, consider a set of vertices of T, $v_1, \ldots, v_{k(m,T)}$, whose removal leaves $p \geq m+k$ components, $T_1, \ldots, T_p$. For each of these components construct a matrix $C_i \in \mathcal{S}(T_i)$ whose smallest eigenvalue is λ with multiplicity 1 (trivial). Let C be any matrix in $\mathcal{S}(T)$ with the submatrices C_i in appropriate positions. By interlacing (theorem 2.1), $m_C(\lambda) \geq p - k \geq m$ and $l_C(\lambda) \leq k(m, T)$. But, since $l_C(\lambda) \geq k(m_C(\lambda), T) \geq k(m, T)$, as well, we have $l_C(\lambda) = k(m, T)$. A construction of C' is analogous. $\square$

Example 3.5 We will compute $k(m, T)$ for several values of m and the tree T,



It is easy to see that $k(1,T) = 0$ and $k(2,T) = 1$.

Because one vertex, namely v_1, can be removed to leave $4 \geq 3+1$ components, $k(3,T) = 1$. The choice of v_1 is intuitive; to maximize the number of components, it is natural to choose the highest degree vertex.

Now consider $k(4,T)$. We already saw that removing v_1 leaves only $4 < 4+1$ components, so $k(4,T) > 1$. Intuition may tell us to continue removing high degree vertices until a sufficient number of components is obtained. However, by inspection, removing v_1 and v_2 leaves $5 < 4+2$ components, and removing v_1, v_2, and v_3 leaves $6 < 4+3$ components. No matter how many more vertices are removed, the conditions for defining $k(4,T)$ will never be satisfied.

Hence, if $k(4,T) < +\infty$, then its value must correspond to a set of vertices that does not include v_1. In fact, removing v_2, v_3, and v_4 leaves $7 \geq 4+3$ components, so $k(4,T) = 3$. (Check that no smaller set of vertices defines $k(4,T)$.) We see that removing v_1 is ineffective because that vertex is adjacent to other high degree vertices.

To calculate $k(m,T)$, sets of vertices must be enumerated in some fashion. Unfortunately, a greedy strategy does not work. The highest degree vertex may not belong to the "winning" set of vertices, and a set of vertices which defines $k(m,T)$ is, in general, unrelated to the set that defines $k(m+1,T)$.

Corollary 3.6 *Let T be a tree on n vertices with degree sequence: $d_1 \geq d_2 \geq \cdots \geq d_n$. If $A \in \mathcal{S}(T)$ and $m = m_A(\lambda) \geq 1$, then*

$$l_A(\lambda) \geq k'(m,T) \text{ and } r_A(\lambda) \geq k'(m,T).$$

Equality is possible if T is segregated and $k(m+1,T) > k(m,T)$.

PROOF. Apply observation 3.2 (part 1) to theorem 3.4. If T is segregated and $k(m+1,T) > k(m,T)$, then the matrix C of theorem 3.4 satisfies $l_C(\lambda) = k(m,T) = k'(m,T)$ and $m_C(\lambda) = m$. (If $m_C(\lambda)$ were greater than m, then $l_C(\lambda)$ would be at least $k(m+1,T) > k(m,T)$.) □

Corollary 3.7 *Let T be a tree in which the maximum degree of a vertex is $d > 2$. If $A \in \mathcal{S}(T)$ and $m = m_A(\lambda)$, then*

$$l_A(\lambda) \geq \frac{m-1}{d-2} \text{ and } r_A(\lambda) \geq \frac{m-1}{d-2}.$$

PROOF. If T has vertex degree sequence $d = d_1 \geq d_2 \geq \cdots \geq d_n$, then for any nonnegative integer k,

$$1 + (d-2)k \geq 1 + \sum_{i=1}^{k}(d_i - 2).$$

Now, $k'(m,T)$ is the least k such that $1 + \sum_{i=1}^{k}(d_i - 2) \geq m$, so

$$1 + (d-2)k'(m,T) \geq 1 + \sum_{i=1}^{k'(m,T)}(d_i - 2) \geq m.$$

Therefore, $k'(m,T) \geq \dfrac{m-1}{d-2}$. □

Corollary 3.8 *If T is a binary tree, $A \in \mathcal{S}(T)$ and $m = m_A(\lambda) \geq 1$, then*

$$l_A(\lambda) \geq m - 1 \text{ and } r_A(\lambda) \geq m - 1.$$

For each binary tree T and each positive integer m that occurs as a multiplicity in some $\sigma(A)$, $A \in \mathcal{S}(T)$, there exist matrices for which equality occurs in the above inequalities.

PROOF. The fact that $l_A(\lambda), r_A(\lambda) \geq m-1$ is a consequence of the previous corollary.

Suppose T is a binary tree, and m a given positive integer. Choose any $m-1$ degree 3 vertices, $v_1, \ldots, v_{m-1}$, such that no two of these vertices are adjacent. (If such a selection is impossible, then $p(T) < m$, a contradiction to theorem 2.5.) There are then $1 + \sum_{i=1}^{m-1}(\deg(v_i) - 1) = 2m - 1$ components in $T - \{v_1, \ldots, v_{m-1}\}$. Because the removal of $m-1$ vertices leaves $2m-1 = m+(m-1)$ components, $k(m,T) \leq m-1$. But $k(m,T) \geq m-1$, so $k(m,T) = m-1$.

Now construct a matrix $C \in S(T)$, as described in the proof of theorem 3.4. This matrix satisfies $m_C(\lambda) \geq m$ and $l_C(\lambda) = m-1$. If $m_C(\lambda) > m$, then $l_C(\lambda) < m_C(\lambda) - 1$, contradicting the first part of this theorem. Therefore, $m_C(\lambda) = m$, and $l_C(\lambda) = m-1$. □

The following has been noted before (e.g. [JD99]), but follows here in quite a different way.

Corollary 3.9 *If T is a tree, the largest and smallest eigenvalues of each $A \in \mathcal{S}(T)$, have multiplicity 1.*

PROOF. Assume T is not a path. Otherwise the result is trivial. If the multiplicity were greater than 1, corollary 3.7 would imply a distinct eigenvalue to the left (right) of the smallest (largest), a contradiction. $\square$

Corollary 3.10 *If T is a tree on at least 3 vertices, and the multiplicity of the second largest (smallest) eigenvalue of $A \in \mathcal{S}(T)$ is m, then there is a vertex of T of degree at least $m+1$.*

PROOF. If T is a path, the result is trivial. Assume T is not a path. Let λ be the second largest eigenvalue. In order that $l_A(\lambda) = 1$, corollary 3.7 says that $1 \geq \dfrac{m-1}{d-2}$, or $d \geq m+1$. $\square$

4 Vertex degrees

It also follows from corollary 3.7 that if the k-th largest eigenvalue $(1 < k < n)$ of an n-by-n $A \in \mathcal{S}(T)$ has multiplicity m, then there is a vertex of T of degree at least $\frac{m+2k-3}{k-1}$.

Corollary 3.10 is the special case $k = 2$. (Of course, the same applies to the k-th smallest eigenvalue by replacing A with $-A$.)

Here, we note that more can be said by taking degrees of additional vertices into account. Again, the bound $d \geq \frac{m+2k-3}{k-1}$ will follow, but it will also be possible to show that the further statements are best possible.

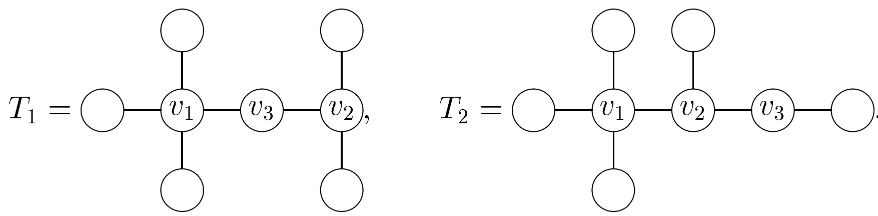
Theorem 4.1 *Let T be a tree on n vertices and $A \in \mathcal{S}(T)$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. If $1 < k < n$ and $m_A(\lambda_k) = m \geq 2$ then, there are $r \leq \min\{k-1, n-k\}$ vertices of T, the sum of whose degrees is at least $m + 2(r+1) - 3 + e$, in which e is the number of edges of the subgraph of T induced by the r vertices. Moreover, a Parter set of vertices for λ_k satisfies these conditions.*

PROOF. There exists a minimal Parter set of vertices $\{v_1, \ldots, v_r\}$ such that $m_{A(v_1,\ldots,v_r)}(\lambda_k) = m + r$ and λ_k is not a multiple eigenvalue of any direct summand of $A(v_1, \ldots, v_r)$. From theorem 2.3, $r \geq 1$, and, by the interlacing inequalities, $r \leq \min\{k-1, n-k\}$. If e is the number of edges in the subgraph of T induced by $\{v_1, \ldots, v_r\}$, then $0 \leq e \leq r-1$, and the number of components in $T - \{v_1, \ldots, v_r\}$ is $1 + \sum_{i=1}^{r}[\deg(v_i) - 1] - e$. But the Parter set was chosen so that $T - \{v_1, \ldots, v_r\}$ must have at least $m + r$ components. Thus, $1 + \sum_{i=1}^{r}[\deg(v_i) - 1] - e \geq m + r$ gives $\sum_{i=1}^{r} \deg(v_i) \geq m + 2(r+1) - 3 + e$. $\square$

Note that, in case $k = 2$, for example, the proof of theorem 4.1 shows that if there is only one vertex of degree at least $m + 1$, it must be a Parter vertex.

If the r vertices from theorem 4.1 are nonadjacent, the sum of the degrees must be at least $m + 2(r + 1) - 3$. If the r vertices form a subtree, the sum of degrees must be at least $m + 3(r + 1) - 5$ since $e = r - 1$. In general, from the degree sequence of a given tree it is not possible to know the structure of the tree in terms of adjacency among vertices. In fact, there are different (nonisomorphic) trees with the same degree sequence. If the only information from the tree is the degree sequence, the best we can say about the degree sum of the r vertices from theorem 4.1 is that it must be, at least, $m+2(r+1)-3$.

Example 4.2 Consider the following two trees on 8 vertices with the same degree sequence $4, 3, 2, 1, 1, 1, 1, 1$:



For $m = 4$ and $k = 3$ we have $\sum_{i=1}^{k-1} \deg(v_i) = 7 \geq m + 2k - 3$. There is a matrix A with $\mathcal{G}(A) = T_1$ and eigenvalues $\lambda_1 \leq \cdots \leq \lambda_8$ such that $m_A(\lambda_3) = 4$ while the same is not the case for any matrix B with graph T_2 ($e = 1$). In fact, the path cover number of T_2 is 3. The matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -2\sqrt{2}$, $\lambda_2 = 1 - 2\sqrt{2}$, $\lambda_3 = \cdots = \lambda_6 = 1$, $\lambda_7 = 2\sqrt{2}$ and $\lambda_8 = 1 + 2\sqrt{2}$.

Corollary 4.3 *If T is a tree on n vertices, $A \in \mathcal{S}(T)$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, $1 < k < n$, and $\lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+m-1}$, then there are $k - 1$ vertices of T, the sum of whose degrees is at least $m + 2k - 3$.*

The conclusion remains valid when k is replaced by $n-(k+m-1)+1$ in the last two expressions involving k.

PROOF. Assume first that $m \geq 2$. Let $v_1, \ldots, v_r$ be the Parter set for λ_k described in theorem 4.1. Order the remaining vertices v_i of T so that $\deg(v_{r+1}) \geq \cdots \geq \deg(v_n)$. If $r = k-1$, then the claim follows immediately from theorem 4.1. Otherwise, $r < k-1$. Suppose that $\sum_{i=1}^{k-1} \deg(v_i) < m + 2k - 3$. Then, theorem 4.1 gives

$$\begin{aligned}\sum_{i=r+1}^{k-1} \deg(v_i) &= \sum_{i=1}^{k-1} \deg(v_i) - \sum_{i=1}^{r} \deg(v_i) \\ &< (m+2k-3) - (m+2(r+1)-3) = 2(k-r-1),\end{aligned}$$

whence $1 = \deg(v_{k-1}) \geq \cdots \geq \deg(v_n) \geq 1$. Thus, $\sum_{i=1}^{n} \deg(v_i) = \sum_{i=1}^{k-1} \deg(v_i) + \sum_{i=k}^{n} \deg(v_i) < (m+2k-3) + (n-k+1) = n+m+k-2$. But the sum of the vertex degrees in T equals twice the number of edges, so $2(n-1) = \sum_{i=1}^{n} \deg(v_i) < n+m+k-2$, which implies the contradiction $n < m+k$.

Assume now that $m = 1$, and order the vertices of T by descending degree. If $\sum_{i=1}^{k-1} \deg(v_i) < m+2k-3 = 2(k-1)$, then $\deg(v_i) = 1$ for $i = k, \ldots, n$ which gives again a contradiction.

Therefore, for $m \geq 1$ and $1 < k < n$, $\sum_{i=1}^{k-1} \deg(v_i) \geq m + 2k - 3$.

To finish, observe that the matrix $-A$ has eigenvalues $\alpha_1 \leq \cdots \leq \alpha_n$ with $\alpha_{n-(k+m-1)+1} = \cdots = \alpha_{n-k+1}$. Then, there are $n-(k+m-1)$ vertices of T, the sum of whose degrees is at least $m + 2(n-(k+m-1)+1) - 3$. $\square$

Example 4.4 Moreover, given integers $m \geq 2$ and $k \geq 2$, there exist a tree T and a matrix $A \in \mathcal{S}(T)$ such that

(1) The sum of the $k-1$ highest vertex degrees in T is $m+2k-3$.
(2) A has an eigenvalue λ such that $m_A(\lambda) = m$ and $l_A(\lambda) = k-1$.

Consider a tree T whose $k-1$ highest degree vertices are $v_1, \ldots, v_{k-1}$, and suppose T satisfies the following: every vertex in $T - \{v_1, \ldots, v_{k-1}\}$ has degree ≤ 2; $v_1, \ldots, v_{k-1}$ are nonadjacent; and $\sum_{i=1}^{k-1} \deg(v_i) = m + 2k - 3$. Because $v_1, \ldots, v_{k-1}$ are nonadjacent, removing these vertices leaves exactly $1 + \sum_{i=1}^{k-1} (\deg(v_i) - 1) = m + (k-1)$ components, $T_1, \ldots, T_{m+(k-1)}$. In order to construct A, for each of these components construct a matrix $A_i \in \mathcal{S}(T_i)$ whose smallest eigenvalue is λ with multiplicity 1. Let A be any matrix in $\mathcal{S}(T)$ with the submatrices A_i in corresponding positions. Then, λ is the smallest eigenvalue of $A(v_1, \ldots, v_{k-1})$ and $m_{A(v_1,\ldots,v_{k-1})}(\lambda) = m + (k-1)$. By the interlacing inequalities, $m_A(\lambda) \geq m$ and $l_A(\lambda) = k-1$. But a path cover argument shows that the maximum multiplicity of any eigenvalue is m [JD99].

5 Two Multiple Eigenvalues

Theorem 5.1 *Let T be a tree and let $A \in \mathcal{S}(T)$ have distinct eigenvalues λ_1 and λ_2. If there exist Parter sets for λ_1 and λ_2 that intersect in at least k vertices, then $b_A(\lambda_1, \lambda_2) \geq k$.*

PROOF. If $\{v_1, \ldots, v_k\}$ is the intersection of the Parter sets, then v_i is a Parter vertex for λ_1 and λ_2 in $A(v_1, \ldots, v_{i-1})$, $i = 1, \ldots, k$. Observing the interlacing inequalites, the number of eigenvalues numerically between λ_1 and λ_2 decreases by 1 each time a "mutual" Parter vertex is removed. Thus, $b_A(\lambda_1, \lambda_2) - k = b_{A(v_1,\ldots,v_k)}(\lambda_1, \lambda_2) \geq 0$. $\square$

The case $k = 1$ in theorem 5.1 yields an immediate corollary. In case there is only one possible Parter vertex, a star (one central vertex from which all other vertices are pendant), it applies immediately whenever there are two or more *multiple* (multiplicity ≥ 2) eigenvalues.

Corollary 5.2 *Let T be a tree and suppose that $A \in \mathcal{S}(T)$ has multiple eigenvalues $\lambda_1 < \lambda_2$ that share a Parter vertex v in T. Then, there is at least one $\lambda \in \sigma(A)$ such that $\lambda_1 < \lambda < \lambda_2$.*

Corollary 5.3 *Let T be a tree in which $d > 2$ is the maximum degree of a vertex, and suppose that A has distinct multiple eigenvalues λ_1 and λ_2. If s is the number of vertices of T of degree at least three, then*

$$b_A(\lambda_1, \lambda_2) \geq \frac{m_A(\lambda_1) + m_A(\lambda_2) - 2}{d - 2} - s.$$

PROOF. From the proof of theorem 3.4 there is a Parter set U for λ_1 and a Parter set V for λ_2 such that every vertex in $U \cup V$ has degree > 2, $\#U \geq k(m_A(\lambda_1), T)$, and $\#V \geq k(m_A(\lambda_2), T)$. By the proof of corollary 3.7, $k(m_A(\lambda_i), T) \geq \frac{m_A(\lambda_i)-1}{d-2}$. So inclusion-exclusion can be applied:

$$\begin{aligned}
\#(U \cap V) &= \#U + \#V - \#(U \cup V) \\
&\geq \frac{m_A(\lambda_1) - 1}{d - 2} + \frac{m_A(\lambda_2) - 1}{d - 2} - \#(U \cup V) \\
&\geq \frac{m_A(\lambda_1) - 1}{d - 2} + \frac{m_A(\lambda_2) - 1}{d - 2} - s \\
&= \frac{m_A(\lambda_1) + m_A(\lambda_2) - 2}{d - 2} - s
\end{aligned}$$

Now apply theorem 5.1. $\square$

The following observation follows rapidly from the definition of Parter vertex.

Observation 5.4 Let T be a tree and suppose that v is a vertex of T such that $\deg(v) = 3$ and one of the neighbors of v is pendant *or* $\deg(v) = 4$ and three of the neighbors of v are pendant. Then, for any $A \in \mathcal{S}(T)$, v is Parter for at most one multiple eigenvalue of A.

We may then identify a class of trees for which observation 5.4 applies to every potential Parter vertex. Call a tree *diametric* if there is a longest path along which all degree ≥ 3 vertices lie. If, further, every vertex is at most one edge from this path, call the tree *depth one*.

Corollary 5.5 *If T is a binary, diametric, depth one tree, then no vertex of T is Parter for more than one multiple eigenvalue of any $A \in \mathcal{S}(T)$.*

For an n-by-n Hermitian matrix A, the *ordered multiplicities* $m_1, m_2, \ldots, m_k$ are the (algebraic) multiplicities of the eigenvalues listed in (ascending) order of the numerical values of the eigenvalues. Of course, $m_1 + \cdots + m_k = n$. For example, if A is a 10-by-10 matrix with eigenvalues -3, -2, -2, 0, 0, 0, 1, 5, 5, 6, then the ordered multiplicities are 1, 2, 3, 1, 2, 1. Because of corollary 5.2, such a multiplicity list cannot occur when T is a star, but corollary 5.2 is essentially the only restriction on the ordered multiplicities for a star, as indicated by our final result.

Theorem 5.6 *Let T be the star on n vertices. There is an $A \in \mathcal{S}(T)$ with ordered multiplicities*

$$m_1, m_2, \ldots, m_k$$

if and only if

(1) $\sum_{i=1}^{k} m_i = n$
and
(2) $m_i > 1 \Longrightarrow 1 < i < k$ *and* $m_{i-1} = 1 = m_{i+1}$.

PROOF. First, the stated conditions are necessary: condition 1 because A is n-by-n and condition 2 because of corollaries 3.9 and 5.2 (only the center vertex can be Parter in a star).

For sufficiency of the stated conditions, first consider the (ordered) multiplicities that exceed 1: $m_{i_1}, m_{i_2}, \ldots, m_{i_p}$, $1 < i_1 < i_2 < \cdots < i_p < k$ and $i_{j+1} - i_j > 1$, $j = 1, \ldots, p-1$. Choose $\lambda_{i_1} < \lambda_{i_2} < \cdots < \lambda_{i_p}$ and let A have $m_{i_j} + 1$ diagonal entries (corresponding to degree 1 vertices) equal to λ_{i_j}, $j = 1, \ldots, p$. In addition, if $i_{j+1} - i_j = \Delta_j > 2$, choose $\Delta_j - 2$ additional distinct "degree 1" diagonal entries between λ_{i_j} and $\lambda_{i_{j+1}}$. If $i_1 > 2$, choose $i_1 - 2$ such distinct diagonal entries $< \lambda_{i_1}$ and if $k - i_p > 1$, choose another

$k - i_p - 1$ such distinct diagonal entries $> \lambda_{i_p}$. This exactly assigns all such diagonal entries; all other entries of the desired matrix may be chosen arbitrarily. The interlacing inequalities insure that the constructed matrix has the desired (ordered) multiplicities. $\square$

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