

# Hermitian Matrices, Eigenvalue Multiplicities, and Eigenvector Components\*

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## Abstract

Given an  $n$ -by- $n$  Hermitian matrix  $A$  and a real number  $\lambda$ , index  $i$  is said to be Parter (resp. neutral, downer) if the multiplicity of  $\lambda$  as an eigenvalue of  $A(i)$  is one more (resp. the same, one less) than that in  $A$ . In case the multiplicity of  $\lambda$  in  $A$  is at least 2 and the graph of  $A$  is a tree, there are always Parter vertices. Our purpose here is to advance the classification of vertices and, in particular, to relate classification to the combinatorial structure of eigenspaces. Some general results are given and then used to deduce some rather specific facts, not otherwise easily observed. Examples are given.

## 1 Introduction

Throughout,  $A$  will be an  $n$ -by- $n$  Hermitian matrix and  $A(i)$  its  $(n - 1)$ -by- $(n - 1)$  principal submatrix, resulting from deletion of row and column  $i$ ,  $i = 1, \dots, n$ . If  $\lambda \in \mathbb{R}$  is an identified eigenvalue, we denote by  $m_A(\lambda)$

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its multiplicity as an eigenvalue of  $A$ . Because of the interlacing inequalities [HJ, Ch. 4],  $|m_A(\lambda) - m_{A(i)}(\lambda)| \leq 1$ , and all 3 values of  $m_A(\lambda) - m_{A(i)}(\lambda)$  are possible. Because of recent work [JLD99, JLD02, JLDSSW, JLDSa] and for historical reasons [P], we call the index  $i$  a *Parter* (resp. *downer*, *neutral*) index if  $m_A(\lambda) - m_{A(i)}(\lambda) = -1$  (resp.  $1, 0$ ). In the event that the graph of  $A$  becomes relevant, recall that  $G(A)$  is the graph on  $n$  vertices in which there is an edge between  $i$  and  $j$  if and only if the  $i, j$  entry of  $A$  is nonzero. By  $\mathcal{H}(G)$  we denote the set of all Hermitian matrices whose graph is the given graph  $G$ ; note that the diagonal entries of the matrices in  $\mathcal{H}(G)$  are immaterial (except that they are real). In discussing issues herein, we naturally identify vertices of  $G(A)$  and indices, and induced subgraphs of  $G(A)$  with principal submatrices of  $A$ , etc., in a benign way. It was shown in [P] and subsequent refinements [W, JLDSa], that *for trees* there are always Parter vertices when  $m_A(\lambda) \geq 2$  and further information about their existence when  $m_A(\lambda) < 2$ . As the location of Parter vertices in  $G$  is an important issue, our purpose here is to relate the classification of vertices (w.r.t. Parter, downer, and neutral) to the combinatorial structure of eigenspaces. However, the relationship may be of interest in both directions. As it turns out some of our observations do not depend upon the particular structure of  $G(A)$ .

If  $m_A(\lambda) \geq 1$ , denote the corresponding eigenspace by  $E_A(\lambda)$ . In the event that entry  $i$  of  $x$  is 0 for every  $x \in E_A(\lambda)$ , we say that  $i$  is a *null vertex* (for  $A$  and  $\lambda$ ); otherwise  $i$  is a *nonzero vertex*. Of course, there is an  $x \in E_A(\lambda)$  whose support consists of all nonzero vertices.

For trees, a useful characterization of when a vertex is Parter was demonstrated in [JLDSa]. Removal of a vertex  $v$  of degree  $d$  in a tree leaves  $d$  induced subgraphs, each of which is a tree; such subgraphs are called branches at  $v$  and may be identified by the neighbors  $u_1, \dots, u_d$  of  $v$ . Vertex  $v$  is then Parter in the tree  $T$  if and only if there is an  $i$ ,  $1 \leq i \leq d$ , such that  $u_i$  is a downer vertex in its branch (all w.r.t. some  $\lambda \in \sigma(A)$ ,  $A \in \mathcal{H}(T)$ ).

We also use the notation  $A(i_1, \dots, i_k)$  to indicate the  $(n - k)$ -by- $(n - k)$  principal submatrix of  $A$  resulting from deletion of rows and columns  $i_1, \dots, i_k$  from  $A \in M_n$ . In addition,  $A[i_1, \dots, i_k]$  denotes the  $k$ -by- $k$  principal submatrix of  $A$  resulting from deletion of all rows and columns *except*  $i_1, \dots, i_k$ . When indices/vertices are deleted, we refer to the remaining vertices via their original numbers.

## 2 General result

From a simple and standard calculation, it is clear that when  $i$  is a null vertex, the structure of  $E_A(\lambda)$  imparts a good deal of information about  $E_{A(i)}(\lambda)$ . Suppose, w.l.o.g., that  $n = i$ :

$$\begin{bmatrix} A(n) & a_{1n} \\ a_{1n}^* & a_{nn} \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x \\ 0 \end{bmatrix}. \quad (2.1)$$

Then,  $A(n)x = \lambda x$ . This implies, in particular, that a null vertex is, at least, neutral. The converse is also valid.

**Theorem 2.1.** *Let  $A$  be an  $n$ -by- $n$  Hermitian matrix. Then, index  $i$  is null for  $A$  if and only if index  $i$  is either Parter or neutral.*

Our proof uses the following lemma. Note that if  $m_A(\lambda) = 0$ , we may, for convenience, adopt the convention that  $E_A(\lambda)$  contains only the zero vector. When taking principal submatrices, it is convenient to think of  $E_{A(i)}(\lambda)$  as a subspace of  $\mathbb{C}^n$ . We define  $E'_{A(i)}(\lambda)$  to be the  $m_{A(i)}(\lambda)$ -dimensional subspace of  $\mathbb{C}^n$  such that the  $i$ th entry of every vector is 0, and deleting the  $i$ th entry from every vector leaves  $E_A(\lambda)$ .

**Lemma 2.2.** *For an  $n$ -by- $n$  Hermitian matrix  $A$  and an identified  $\lambda \in \mathbb{R}$ , we have*

- i) If  $i$  is downer, then  $E_A(\lambda) \supsetneq E'_{A(i)}(\lambda)$ .*
- ii) If  $i$  is neutral, then  $E_A(\lambda) = E'_{A(i)}(\lambda)$ .*
- iii) If  $i$  is Parter, then  $E_A(\lambda) \subsetneq E'_{A(i)}(\lambda)$ .*

*Proof.* Assume w.l.o.g. that  $i = n$  and  $\lambda = 0$ , and use the block decomposition of  $A$  shown in (2.1).

If  $a_{1n}^*$  is a linear combination of the rows of  $A(n)$ , then  $E_A(0) \supseteq E'_{A(n)}(0)$ . If  $a_{1n}^*$  is not a linear combination of the rows of  $A(n)$ , then  $\text{rank } A = \text{rank } A(n) + 2$  (since  $A$  is Hermitian), so  $n$  is Parter. Hence, if  $n$  is downer or neutral,  $E_A(0) \supseteq E'_{A(n)}(0)$ . By definition, if  $n$  is downer, the containment is strict, and if  $n$  is neutral, the containment is actually equality.

Suppose  $n$  is Parter. Let  $X$  be the maximal subspace of  $E'_{A(i)}$  that is orthogonal to  $(a_{1n}^* \ 0)$ . Clearly,  $X \subseteq E_A(\lambda)$ . Since  $\dim X \geq m_{A(i)}(\lambda) - 1 = m_A(\lambda)$ , we have  $X = E_A(\lambda)$ .  $\square$

*Proof of Theorem 2.1.* Return to the calculation displayed in (2.1). Index  $i$  is null for  $A$  if and only if  $E_A(\lambda) \subseteq E'_{A(i)}(\lambda)$ . By the lemma, this is true if and only if  $i$  is Parter or neutral.  $\square$

### 3 Distinguishing Parter and neutral vertices

To distinguish between Parter and neutral vertices, then, we must look beyond the appropriate eigenspace of  $A$  itself. Our approach is to consider the secondary eigenspace, that of  $A(i)$ , associated with the same  $\lambda$ . We continue to write  $A$  as a block matrix as in (2.1). Again, we begin with some useful lemmas.

**Lemma 3.1.** *If  $n$  is a null vertex, then  $n$  is neutral if and only if  $E_{A(n)}(\lambda)$  is orthogonal to  $a_{1n}^*$ .*

*Proof.* By Lemma 2.2,  $E_A(\lambda) \subseteq E'_{A(n)}(\lambda)$ . In fact,  $E_A(\lambda)$  is precisely the maximal subspace of  $E'_{A(n)}(\lambda)$  that is orthogonal to  $(a_{1n}^* \ 0)$ . Thus,  $n$  is neutral if and only if  $E_A(\lambda) = E'_{A(n)}(\lambda)$  if and only if  $E_{A(n)}(\lambda)$  is orthogonal to  $a_{1n}^*$ .  $\square$

There is a particularly simple sufficient condition for orthogonality. We say that a subspace  $X \subseteq \mathbb{C}^n$  is *combinatorially orthogonal* to a vector  $y \in \mathbb{C}^n$

if  $x_i \bar{y}_i = 0$  for every  $x \in X, i = 1, \dots, n$ .

**Lemma 3.2.** *Suppose that the graph of  $A$  is a tree, and that  $n$  is a null vertex. The following statements are equivalent.*

- i)  $n$  is neutral.*
- ii) All neighbors of  $n$  are null for  $A(n)$ .*
- iii)  $E_{A(n)}(\lambda)$  is orthogonal to  $a_{1n}^*$ .*
- iv)  $E_{A(n)}(\lambda)$  is combinatorially orthogonal to  $a_{1n}^*$ .*

*Proof.*  $1 \Rightarrow 2$ : The vertex  $i$  is Parter if and only if some neighbor of  $i$  is a downer vertex in  $A(i)$ . Downer vertices coincide with nonzero vertices by Theorem 2.1.

$2 \Rightarrow 4$ : The only nonzero entries in  $a_{1n}^*$  correspond to the neighbors of  $i$ . These neighbors are null vertices by assumption.

$4 \Rightarrow 3$  is obvious.

$3 \Rightarrow 1$  by Lemma 3.1. □

We may now state our main observation of this section. It identifies Parter vertices among null vertices by considering eigenspaces of  $A(i)$ .

**Theorem 3.3.** *Let  $A$  be an Hermitian matrix whose graph is a tree, and let  $i$  be a null vertex for  $A$ . Then  $i$  is Parter if and only if there is a neighbor  $j$  that is nonzero for  $A(i)$ .*

*Proof.* This is the contrapositive of  $1 \Leftrightarrow 2$  of Lemma 3.2. We can also prove the result directly. The vertex  $i$  is Parter if and only if some neighbor of  $i$  is a downer vertex in its branch of  $G(A) \setminus i$ . If such a downer vertex exists, then it is nonzero for  $A(i)$ . If not, then every neighbor of  $i$  is null for  $A(i)$ .  $\square$

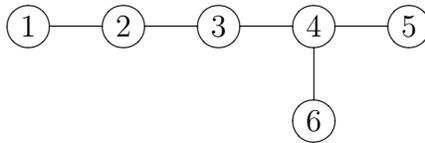
Our theorem has a surprising corollary.

**Corollary 3.4.** *Suppose that the graph of  $A$  is a tree. Every neighbor of a neutral vertex is a null vertex for  $A$ .*

*Proof.* By the theorem, if  $i$  is neutral, then every neighbor of  $i$  is null for  $A(i)$ . Because  $E_A(\lambda) = E'_{A(i)}(\lambda)$ , every vertex that is null for  $A(i)$  is also null for  $A$ .  $\square$

The corollary implies, by Theorem 2.1, that every neighbor of a neutral vertex is either neutral or Parter. Thus, if a null vertex is not Parter, its neighbors constitute a natural place to look for Parter vertices. It can happen that all neighbors are again neutral, but, often, the neighbors include a Parter vertex.

*Example 3.5.* The converse of Corollary 3.4 is not true. It may happen that all neighbors of a null vertex are null without the vertex being neutral. Suppose an Hermitian matrix  $A$  with graph



satisfies  $m_{A[1,2]}(\lambda) = m_{A[5]}(\lambda) = m_{A[6]}(\lambda) = 1$ ,  $m_{A[3]}(\lambda) = 0$ . Then vertex 4 is Parter because vertex 5 is a downer for its branch. A consequence of the discussion on paths following this example is that  $\lambda$  is not an eigenvalue of  $A[1, 2, 3]$ . Hence,  $m_A(\lambda) = 1$ . Now it is easy to check that vertex 2 is neutral and vertices 3 and 4 are Parter.  $\square$

Let  $A$  be an Hermitian matrix whose graph is a path, and let  $\lambda$  be an eigenvalue of  $A$ . For example,  $A$  could be an irreducible, tridiagonal Hermitian matrix. Using Theorem 2.1, we can locate the zeros in an eigenvector corresponding to  $\lambda$ . We begin by classifying the possible locations of Parter, neutral, and downer vertices. It is a well known fact that deleting a pendant (i.e. degree 1) vertex from  $A$  causes the eigenvalue interlacing inequalities to be strict, and thus each pendant vertex is a downer vertex for  $\lambda$ . It follows that  $A$  has no neutral vertices, because if  $\lambda$  is an eigenvalue of  $A(i)$ , each neighbor of  $i$  is a pendant vertex in  $G(A(i))$ , and thus a downer vertex for its branch, forcing  $i$  to be Parter. For the same reason, if  $i$  and  $j$  are Parter vertices, then  $j$  is Parter for  $A(i)$ , and hence no two Parter vertices can be adjacent. The converse of these three observations is also true; specifically, if  $i_1 \leq \dots \leq i_k$  satisfy  $i_1 \neq 1$ ,  $i_k \neq n$ , and  $i_{j+1} - i_j > 1$  for  $j = 1, \dots, k - 1$ , then there exists an irreducible, tridiagonal Hermitian matrix for which  $\lambda$  is an eigenvalue and vertices  $i_1, \dots, i_k$  are precisely the Parter vertices. (Simply construct a  $B$  such that  $\lambda$  is an eigenvalue of each direct summand of  $B(i_1, \dots, i_k)$  and  $B(i_1, \dots, i_k)$  has no Parter vertices (trivial).) Furthermore, if  $\lambda$  is the  $r$ th largest (resp. smallest) eigenvalue of  $A$ , then  $\lambda$  can have at

most  $r - 1$  Parter vertices, i.e.  $k \leq r - 1$ . (To see this, iterate the interlacing inequalities to see that  $\lambda$  is the  $(r - k)$ th largest (resp. smallest) eigenvalue of  $m_{A(i_1, \dots, i_k)}(\lambda)$ .) Now, by Theorem 2.1, the constraints on  $i_1, \dots, i_k$  also characterize the locations of zeros in an eigenvector.

## 4 Implications

The observations made thus far show that there are simple but surprising links among the classification of vertices. These have some very strong implications that we explore here. First, we give another basic lemma that holds independent of the graph and then consider implications via certain categories.

**Lemma 4.1.** *Let  $A$  be an  $n$ -by- $n$  Hermitian matrix. If  $i$  is neutral, then  $j \neq i$  is downer for  $A$  if and only if  $j$  is downer for  $A(i)$ .*

*Proof.* If  $i$  is neutral, then  $E_A(\lambda) = E'_{A(i)}(\lambda)$ , which implies that  $j$  is nonzero for  $A$  if and only if  $j$  is nonzero for  $A(i)$ .  $\square$

### 4.1 Vertex classification

It is a goal for each graph  $G$ ,  $A \in \mathcal{H}(G)$ , and identified  $\lambda \in \mathbb{R}$ , to be able to quickly classify each vertex w.r.t. Parter, neutral, or downer. In principle this could be done with prior results. Here, we mention some observations that assist in classification.

**Proposition 4.2.** *Let  $A$  be an  $n$ -by- $n$  Hermitian matrix. If  $m_A(\lambda) = m$ , then  $A$  has at least  $m$  downer vertices.*

*Proof.* Assume  $m \geq 1$ . Because  $\dim E_A(\lambda) = m$ , there is some vector in  $E_A(\lambda)$  that has at least  $m$  nonzero entries. These entries identify downer vertices.  $\square$

**Proposition 4.3.** *Suppose that the graph of  $A$  is connected. If  $m_A(\lambda) = m \geq 1$ , then  $A$  has at least  $m + 1$  downer vertices.*

*Proof.* By Proposition 4.2,  $A$  has at least  $m$  nonzero vertices. Suppose  $A$  has exactly  $m$  nonzero vertices. Then  $E_A(\lambda)$  is spanned by vectors  $e_{i_1}, \dots, e_{i_m}$ , where  $e_j$  is the  $j$ th standard basis vector for  $\mathbb{C}^n$ . Since  $(A - \lambda I)e_j = 0$  implies the  $j$ th column of  $A - \lambda I$  is zero, the graph of  $A$  is not connected.  $\square$

*Example 4.4.* A *star* is a graph that is a tree and has exactly one vertex of degree  $> 1$ . If the graph of  $A$  is the star on  $n$  vertices, and every diagonal entry of  $A$  is  $\lambda$ , then  $m_A(\lambda) = n - 2$ . Also, the central vertex is Parter, and every pendant vertex is a downer vertex, so  $A$  has exactly  $m_A(\lambda) + 1$  downer vertices. Therefore, Proposition 4.3 is the strongest statement that can be made for all connected graphs.  $\square$

The following proposition is a restatement of Corollary 3.4.

**Proposition 4.5.** *Suppose that the graph of  $A$  is a tree, and let  $i$  be a neutral vertex. Then every neighbor of  $i$  is either Parter or neutral for  $A$ .*

## 4.2 Classification of vertex pairs

We next turn to the classification of two vertices and, in particular, the possibilities for their status initially vs. sequentially. There are differences depending upon whether or not the two vertices are adjacent. We begin with another observation that is independent of the graph.

**Proposition 4.6.** *Let  $A$  be an  $n$ -by- $n$  Hermitian matrix, and let  $i$  and  $j$  be distinct indices. We have the following two statements.*

*i) If  $i$  and  $j$  are neutral, then  $m_A(\lambda) - m_{A(i,j)}(\lambda) \in \{-1, 0\}$ .*

*ii) If  $i$  is neutral and  $j$  is downer, then  $m_A(\lambda) - m_{A(i,j)}(\lambda) = 1$ .*

*Proof.* By Lemma 4.1, if  $i$  and  $j$  are neutral, then  $j$  is Parter or neutral for  $A(i)$ .

By the same lemma, if  $i$  is neutral and  $j$  is downer, then  $j$  is downer for  $A(i)$ . □

**Proposition 4.7.** *Suppose that the graph of  $A$  is a tree  $T$ , and let  $i$  and  $j$  be distinct indices. If  $i$  and  $j$  are Parter, then  $m_A(\lambda) - m_{A(i,j)}(\lambda) \in \{-2, 0\}$ .*

*Proof.* Clearly,  $m_{A(i,j)}(\lambda) - 2 \leq m_A(\lambda) \leq m_{A(i,j)}(\lambda)$ . Suppose  $m_A(\lambda) = m_{A(i,j)}(\lambda) - 1$ . Then  $i$  is neutral in  $A(j)$ , and  $j$  is neutral in  $A(i)$ . This implies that exactly one neighbor  $k$  of  $i$  is a downer vertex in its branch, and that this branch contains  $j$ . Call this branch  $T_1$ . If we remove  $i$ , then we find that  $k$  is downer in  $T_1$  and  $j$  is neutral in  $T_1$ . By Proposition 4.6,  $m_{A(i,j,k)}(\lambda) =$

$m_{A(i)}(\lambda) - 1 = m_A(\lambda)$ . But  $m_{A(i,j)}(\lambda) = m_A(\lambda) + 1$  by assumption, so  $k$  is a downer vertex for  $A(i, j)$ . Since  $k$  is adjacent to  $i$ , it follows that  $i$  is Parter for  $A(j)$ . Therefore,  $m_A(\lambda) = m_{A(i,j)}(\lambda) - 2$ , a contradiction.  $\square$

**Corollary 4.8.** *Suppose that the graph of  $A$  is a tree, and let  $i$  and  $j$  be distinct indices. If  $i$  is Parter and  $m_A(\lambda) - m_{A(i,j)}(\lambda) = -1$ , then  $j$  is neutral for  $A$ .*

*Proof.* By Proposition 4.7, if  $j$  is Parter, then  $m_A(\lambda) - m_{A(i,j)}(\lambda) \neq -1$ . If  $j$  is downer, then  $m_{A(i,j)}(\lambda) \leq m_{A(j)}(\lambda) + 1 = m_A(\lambda)$ .  $\square$

**Proposition 4.9.** *Suppose that the graph of  $A$  is a tree, and let  $i$  and  $j$  be neighbors. We have the following two statements.*

- i) If  $i$  and  $j$  are neutral, then  $m_A(\lambda) - m_{A(i,j)}(\lambda) = 0$ .*
- ii) If  $i$  and  $j$  are downer, then  $m_A(\lambda) - m_{A(i,j)}(\lambda) = 1$ .*

*Proof.* By Proposition 4.6, if  $i$  and  $j$  are neutral, then  $m_A(\lambda) - m_{A(i,j)}(\lambda) \in \{-1, 0\}$ . Suppose  $m_A(\lambda) - m_{A(i,j)}(\lambda) = -1$ . Then  $j$  is Parter in  $A(i)$ , so  $j$  is adjacent to a vertex  $k$  which is downer for  $A(i, j)$ . But then  $k$  must also be a downer in  $A(j)$  since  $i$  and  $j$  are adjacent. It follows that  $j$  is Parter for  $A$ , a contradiction.

If  $i$  and  $j$  are downer, then clearly  $0 \leq m_A(\lambda) - m_{A(i,j)}(\lambda) \leq 2$ . Suppose that  $m_A(\lambda) - m_{A(i,j)}(\lambda) = 0$ . Then  $j$  is Parter in  $A(i)$ , so  $j$  is adjacent to some vertex  $k$  which is downer for  $A(i, j)$ . But since  $i$  and  $j$  are adjacent,  $k$  must also be downer for  $A(j)$ , which implies that  $j$  is Parter for  $A$ , a contradiction.

Now suppose that  $m_A(\lambda) - m_{A(i,j)}(\lambda) = 2$ . Then  $j$  is downer for its branch at  $i$ , which implies that  $i$  is Parter for  $A$ , a contradiction.  $\square$

*Example 4.10.* We will show that if  $i$  and  $j$  are not adjacent, then the conclusions of Proposition 4.9 may not hold.

First, observe that if an irreducible 2-by-2 Hermitian matrix has  $\lambda$  on its diagonal, then  $\lambda$  is not an eigenvalue.

Take  $\lambda = 0$ , and let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check that  $m_A(0) = 0$ , and that the graph of  $A$  is a path. Removing either pendant vertex leaves a 2-by-2 Hermitian matrix with  $\lambda = 0$  on its diagonal, so both pendant vertices are neutral. However,  $m_{A(1,3)}(0) = 1$ , so claim 1 of Proposition 4.9 does not hold.

Still with  $\lambda = 0$ , take

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$B$  has the same graph as  $A$ , but  $m_B(0) = 1$ . For the same reason as above,  $m_{B(1)}(0) = m_{B(3)}(0) = 0$ , so the pendant vertices are downer vertices. However, in contrast to claim 2 of Proposition 4.9,  $m_B(0) - m_{B(1,3)}(\lambda) = 0$ .

Again with  $\lambda = 0$ , take

$$C = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then  $m_C(0) = 2$  and  $m_{C(1,2)}(0) = 0$ , so claim 2 of Proposition 4.9 does not hold.  $\square$

Using the results thus far, we are able to classify, for pairs of vertices, the joint or sequential effect upon multiplicity, given the individual effect of removal. It is of interest that certain possibilities cannot occur. Some of these are contained in results of this section thus far, and others are straightforward. We list the full classification without proof. Both the case of arbitrary and adjacent vertices are considered. In each case, a missing possibility provably cannot occur, and examples may be constructed for each listed possibility. For example, the last entry in the second table indicates that if a downer vertex is removed, an adjacent vertex that was initially a downer could then be a downer or neutral, but not Parter (which can occur in the non-adjacent case.)

*Table 4.11.* Suppose that the graph of  $A$  is a tree, and let  $i$  and  $j$  be distinct vertices.

- i) Every value for  $m_A(\lambda) - m_{A(i,j)}(\lambda)$  listed in the following table is attainable, and no other values are.

$i$	$j$	$m_A(\lambda) - m_{A(i,j)}(\lambda)$
Parter	Parter	-2, 0
Parter	Neutral	-1, 0
Parter	Downer	0
Neutral	Neutral	-1, 0
Neutral	Downer	1
Downer	Downer	0, 1, 2

ii) Now suppose that  $i$  and  $j$  are neighbors. Every value for  $m_A(\lambda) - m_{A(i,j)}(\lambda)$  listed in the following table is attainable, and no other values are.

$i$	$j$	$m_A(\lambda) - m_{A(i,j)}(\lambda)$
Parter	Parter	-2, 0
Parter	Neutral	-1, 0
Parter	Downer	0
Neutral	Neutral	0
Neutral	Downer	not possible
Downer	Downer	1

Of course, the particular tree can place further restrictions on  $m_A(\lambda) - m_{A(i,j)}(\lambda)$ . The listed possibilities exist for *some* Hermitian matrix with *some* graph that is a tree.

[FIX non-trees?]

### 4.3 Null subgraphs

This section is devoted to unexpected results about classification of entire subgraphs of a given graph.

Let  $G_1$  be an induced subgraph of  $G(A)$ . If  $x_i = 0$  for all  $x \in E_A(\lambda)$ ,  $i \in G_1$ , then we say that  $G_1$  is a *null subgraph* (for  $A$  and  $\lambda$ ). Our first observation is a simple consequence of Theorem 2.1.

**Proposition 4.12.**  *$G_1$  is a null subgraph for  $A$  if and only if every vertex  $i$  of  $G_1$  is Parter or neutral for  $A$ .*

If there is a sequence of vertices  $i_1, \dots, i_k$  such that  $i_1$  is null for  $A$  and  $i_j$  is null for  $A(i_1, \dots, i_{j-1})$ ,  $j = 2, \dots, k$ , then we say that  $i_1, \dots, i_k$  are *sequentially null*.

**Proposition 4.13.** *Let  $A$  be an  $n$ -by- $n$  Hermitian matrix, and suppose  $i_1, \dots, i_k$  are sequentially null. If  $\lambda$  is not an eigenvalue of some direct summand  $A_1$  of  $A(i_1, \dots, i_k)$ , then  $G(A_1)$  is a null subgraph for  $A$ .*

*Proof.* Clearly, every vertex in  $G(A_1)$  is a null vertex for  $A(i_1, \dots, i_k)$ . For each  $j$ , the eigenspace of  $A(i_1, \dots, i_{j-1})$  is contained in the eigenspace of  $A(i_1, \dots, i_j)$  (by Lemma 2.2), so null vertices of  $A(i_1, \dots, i_j)$  are null vertices of  $A(i_1, \dots, i_{j-1})$ . □

**Proposition 4.14.** *Let  $A$  be an  $n$ -by- $n$  Hermitian matrix, and suppose  $i_1, \dots, i_k$  are sequentially null. Identify some direct summand  $A_1$  of  $A(i_1, \dots, i_k)$ . If  $G(A_1)$  is a null subgraph for  $A$ , then  $m_{A_1}(\lambda) \leq m_{A(i_1, \dots, i_k)}(\lambda) - m_A(\lambda)$ .*

*Proof.* We have  $E_A(\lambda) \subseteq E'_{A(i_1, \dots, i_k)}(\lambda)$ . (Similar to the notation introduced before Lemma 2.2,  $E'_{A(i_1, \dots, i_k)}(\lambda)$  is formed from  $E_{A(i_1, \dots, i_k)}(\lambda)$  by inserting zeros into appropriate spaces.) If  $A(i_1, \dots, i_k) = A_1 \oplus A_2$ , then  $E_A(\lambda) \subseteq E'_{A_2}(\lambda)$ . Now,  $\dim E_{A_1}(\lambda) = \dim E_{A(i_1, \dots, i_k)}(\lambda) - \dim E_{A_2}(\lambda) \leq \dim E_{A(i_1, \dots, i_k)}(\lambda) - \dim E_A(\lambda)$ .  $\square$

**Proposition 4.15.** *Suppose that the graph of  $A$  is a tree. Let  $i$  be Parter for  $A$ , and identify some branch  $T'$  at  $i$  for which  $m_{A[T']}(\lambda) \geq 1$ . If  $T'$  is a null subgraph for  $A$ , then every neighbor of  $i$  is Parter or neutral for  $A$ .*

*Proof.* If  $j$  is a neighbor of  $i$ , let  $T_j$  denote the branch of  $j$  at  $i$ .

In the notation of Lemma 2.2, we have  $E_A(\lambda) \subseteq E'_{A(i)}(\lambda)$ . Choose a basis  $B$  for  $E'_{A(i)}(\lambda)$  in which the support of any basis vector is contained in a single branch  $T_j$ . Because  $m_A(\lambda) - m_{A(i)}(\lambda) = -1$  and  $T'$  is a null subgraph, it follows that  $B$  contains exactly one vector  $x_1$  whose support is  $T'$ . Furthermore,  $B \setminus x_1$  is a basis for  $E_A(\lambda)$ .

Obviously, if there is a neighbor  $j$  such that no basis vector  $x \in B \setminus x_1$  has support  $T_j$ , then  $j$  is a null vertex.

Now, let  $j$  be a neighbor of  $i$ , and suppose there exists an  $x \in B \setminus x_1$  whose support is  $T_j$ . We have  $(A - \lambda I_n)x = 0$ , and  $a_{ij}x_j = 0$  implies  $x_j = 0$ .  $\square$

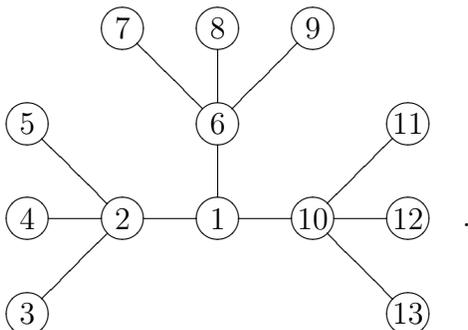
## 5 Example

Results from the previous section can be used to classify vertices w.r.t. Parter, neutral, or downer with little knowledge of the numerical entries in a matrix.

Of course, an understanding of the combinatorial structure of eigenspaces follows. Here we present an extended example.

Several results in the previous section concern the quantity  $m_A(\lambda) - m_{A(i,j)}(\lambda)$ . Sometimes it is useful to think of computing  $A(i, j)$  by first deleting row and column  $i$  and then deleting row and column  $j$ . For example, if  $i$  is Parter for  $A$  and  $j$  is neutral for  $A(i)$ , then  $m_A(\lambda) - m_{A(i,j)}(\lambda) = -1$ . In this case, we say that  $i$  and  $j$  are *sequentially Parter-neutral* (for  $A$  and  $\lambda$ ). We may rephrase Corollary 4.8, “If  $i$  and  $j$  are sequentially Parter-neutral for  $A$ , then  $j$  is *originally* neutral for  $A$ .”

Let  $A = (a_{ij})$  be an Hermitian matrix with graph



Let  $B = A[2, 3, 4, 5]$ ,  $C = A[6, 7, 8, 9]$ , and  $D = A[10, 11, 12, 13]$ . The graph of each of these principal submatrices is the star on four vertices. Let  $\lambda$  be a fixed real number, and suppose that  $m_B(\lambda) = 0$ ,  $m_C(\lambda) = 1$ , and  $m_D(\lambda) = 2$ . Further, suppose that  $C$  has no Parter vertices. We will use this information to classify some of the vertices of  $A$  w.r.t. Parter, neutral, or downer, and to completely classify the combinatorial structure of the eigenspace corresponding to  $\lambda$ .

Because  $m_D(\lambda) = 2$ , it follows that  $D$  has a Parter vertex and  $a_{11,11} = \lambda$ .

Therefore, vertex 11 is a downer for its branch at vertex 10, so vertex 10 is Parter for  $A$ .

Similarly, vertex 6 is downer for  $C$ , so vertex 1 is Parter for  $A$ .

Because vertices 1 and 10 are sequentially Parter-Parter and  $m_{A(1,10)}(\lambda) = 4$ , we conclude  $m_A(\lambda) = 2$ .

By Proposition 4.13, the subgraph of  $G(A)$  induced by vertices 2, 3, 4, 5 is a null subgraph, i.e. each vertex  $i$ ,  $i = 2, 3, 4, 5$ , is Parter or neutral for  $A$ .

Because vertices 1 and 2 are sequentially Parter-neutral, vertex 2 must be neutral for  $A$  by Corollary 4.8.

Vertex 6 is not a downer vertex, because  $m_A(\lambda) - m_{A(6,10)}(\lambda) \leq -1$ , which implies  $m_A(\lambda) - m_{A(6)}(\lambda) \leq 0$ . Similar arguments show that vertices 7, 8, and 9 are Parter or neutral.

By Proposition 4.3,  $A$  has at least three downer vertices, so vertices 11, 12, and 13 must be downers for  $A$ .

In summary,  $m_A(\lambda) = 2$ ; vertices 1 and 10 are Parter; vertex 2 is neutral; vertices 11, 12, and 13 are downer; and each vertex  $i$ ,  $i = 3, 4, 5, 6, 7, 8, 9$ , is either Parter or neutral. Therefore,  $x_i = 0$  for all  $x \in E_A(\lambda)$ ,  $i \neq 11, 12, 13$ , and there is some eigenvector which is nonzero in coordinates 11, 12, and 13.

*Remark 5.1.* Once we know that vertex 2 is neutral, Proposition 4.5 implies that vertices 3, 4, and 5 are null vertices, which agrees with our findings.

*Remark 5.2.* Once we know that vertices 6, 7, 8, and 9 are null vertices, Proposition 4.15 implies that vertices 2 and 10 are also null vertices, which agrees with our findings.

*Remark 5.3.* The constraints on  $A$  were insufficient to classify every vertex as Parter, neutral, or downer. For example, there is a matrix that satisfies the constraints on  $A$  such that vertex 3 is Parter but vertex 4 is neutral, and vertex 7 is Parter but vertex 8 is neutral. It is also possible to show that vertex 6 may be either neutral or Parter. However, if vertex 6 is Parter, then vertices 7, 8, and 9 must be neutral, because then vertices 6 and  $i$ ,  $i = 7, 8, 9$ , would be sequentially Parter-neutral.

## References

- [HJ] Roger A. Horn and Charles R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1985, 1990.
- [JLD99] C. R. Johnson and António Leal Duarte. The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. *Linear and Multilinear Algebra*, 46(1-2):139–144, 1999.
- [JLD02] Charles R. Johnson and António Leal Duarte. On the possible multiplicities of the eigenvalues of a Hermitian matrix whose graph is a tree. *Linear Algebra Appl.*, 348:7–21, 2002.
- [JLDSa] Charles R. Johnson, António Leal Duarte, and Carlos M. Sa-  
iago. The Parter-Wiener theorem: refinement and generalization. *SIAM J. Matrix Anal. Appl.*, 25:352–361, 2003.

- [JLDSSW] Charles R. Johnson, António Leal Duarte, Carlos M. Saiago, Brian D. Sutton, and Andrew J. Witt. On the relative position of multiple eigenvalues in the spectrum of an Hermitian matrix with a given graph. *Linear Algebra Appl.*, 363:147–159, 2003.
- [P] S. Parter. On the eigenvalues and eigenvectors of a class of matrices. *J. Soc. Indust. Appl. Math.*, 8:376–388, 1960.
- [W] Gerry Wiener. Spectral multiplicity and splitting results for a class of qualitative matrices. *Linear Algebra Appl.*, 61:15–29, 1984.

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