Implicit Construction of Multiple Eigenvalues for Trees*

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September 3, 2007

Abstract

We are generally concerned with the possible lists of multiplicities for the eigenvalues of a real symmetric matrix with a given graph. Many restrictions are known, but it is often problematic to construct a matrix with desired multiplicities, even if a matrix with such multiplicities exists. Here, we develop a technique for construction using the implicit function theorem in a certain way. We show that the technique works for a large variety of trees, give examples and determine all possible multiplicities for a large class of trees for which this was not previously known.

AMS Subject Classification: 15A18, 15A42, 05C05
Keywords: eigenvalue, multiplicity, graph, tree

1 Introduction

Let G be an undirected graph on n vertices. A real symmetric matrix $A = (a_{ij})$ is said to have graph G provided $a_{ij} \neq 0$ if and only if vertices i and j

*Research supported by NSF-REU grant DMS-99-87803.

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are adjacent in G; G imposes no restriction upon the diagonal entries of A—they may be 0 or any nonzero real numbers. By $S(G)$ we mean the set of all n-by-n real symmetric matrices whose graph is (precisely) G. We have been interested in the possible spectra among matrices in $S(G)$ and, in particular, the multiplicities that can occur. By *ordered multiplicities* for a symmetric matrix A, we mean the list of multiplicities of the distinct eigenvalues of A, ordered according to the (increasing) order of the numerical values of the underlying eigenvalues. For example, if $n = 10$ and A has eigenvalues: $-3, -1, -1, 0, 2, 2, 2, 3, 3, \pi$, then the ordered multiplicities are

$$1, 2, 1, 3, 2, 1.$$

By *unordered multiplicities*, we simply mean the list of multiplicities in which the values of the eigenvalues are ignored but the multiplicities are listed in non-increasing order; for example the unordered multiplicities version of the above list is

$$3, 2, 2, 1, 1, 1.$$

Sometimes we will speak of the ordered (resp. unordered) multiplicities of a graph G. By this we mean the set of ordered (resp. unordered) multiplicity lists occurring among all matrices in $S(G)$.

In prior work, all possible lists of multiplicities for certain graphs have been determined [JLDSSW, JLDSb]. In that work, constructions were carried out, either by explicit matrix construction or by explicit manipulation of polynomials. Often such constructions are complicated and the difficulty of producing them limits the graphs for which multiplicities can be understood.

2 Background

In previous work, we have seen that the graph of a Hermitian matrix can restrict the ordered multiplicities of the eigenvalues. Prior work has focused on graphs that are *trees*, i.e. connected graphs with no cycles. In this case, it suffices to consider real symmetric matrices, as we do here; every Hermitian matrix whose graph is a tree is diagonally unitarily similar to a real symmetric matrix with the same graph. The following three theorems are useful negative results. We assume that A is a symmetric matrix whose graph is a tree T.

Theorem 2.1 ([JLD99]). *We have*

$$\max_{\lambda \in \sigma(A),\, A \in S(T)} m_A(\lambda) = p(T),$$

in which $m_A(\lambda)$ *is the multiplicity of* λ *as an eigenvalue of* A *and* $p(T)$ *is the* path cover number *of* T*, i.e. the minimum cardinality of any set of vertex-disjoint paths of* T *that cover the vertices of* T*.*

Theorem 2.2 ([LDJ]). *A has at least $d(T)$ distinct eigenvalues, where $d(T)$ is the* diameter *of T, i.e. the number of vertices in a longest path in T.*

Theorem 2.3 ([JLDSSW]). *If T is a binary tree (all vertex degrees are ≤ 3), and $m_A(\lambda) = k$, then A has at least $k-1$ eigenvalues less than λ and at least $k-1$ eigenvalues greater than λ (each counting multiplicities).*

A *star* is a tree with a single central vertex surrounded by several degree 1 vertices. In [JLDSSW], it was shown that a star has only a handful of possible ordered multiplicity lists and then these lists were constructed explicitly. Thus, the ordered multiplicities problem for stars is completely solved.

We would like to implement a similar program—eliminating particular multiplicity lists and constructing the remaining ones—on larger classes of trees, but construction of matrices with given eigenvalue multiplicities is a challenge. Here, we present a new construction technique, based on the implicit function theorem. First, we record two more facts that will prove useful.

Deleting the ith row and column of A naturally corresponds to deleting the ith vertex from the graph of A. Sometimes we will even say that we delete the ith "vertex" of A to form the principal submatrix $A(i)$. The *eigenvalue interlacing inequalities* [HJ, Ch. 4] relate the eigenvalues of the original matrix to the eigenvalues of the principal submatrix. Specifically, suppose that A is symmetric with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, and that $A(i)$ has eigenvalues $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$. Then the μ's *interlace* the λ's, i.e. $\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$.

The interlacing inequalities imply that

$$m_A(\lambda) - m_{A(i)}(\lambda) \in \{-1, 0, 1\}.$$

Thus, to verify $m_A(\lambda) \geq k$, it suffices to exhibit an i for which $m_{A(i)}(\lambda) \geq k+1$.

The implicit function theorem will allow us to perturb some entries of a matrix from zero to nonzero, modifying the graph, while preserving certain eigenvalue multiplicities. A common statement of the theorem is the following.

Theorem 2.4 (Implicit Function Theorem [R])**.** *Let* $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ *be a continuously differentiable function. Suppose that, for* $x_0 \in \mathbb{R}^n$ *and* $y_0 \in \mathbb{R}^m$, $f(x_0, y_0) = 0$ *and the Jacobian* $\det(\partial f/\partial x)(x_0, y_0) \neq 0$*. Then there exists a neighborhood* $U \subset \mathbb{R}^m$ *around* y_0 *such that* $f(x, y) = 0$ *has a solution* x *for any fixed* $y \in U$*. Furthermore, there is a solution* x *arbitrarily close to* x_0 *associated with a* y *sufficiently close to* y_0*.*

3 An example

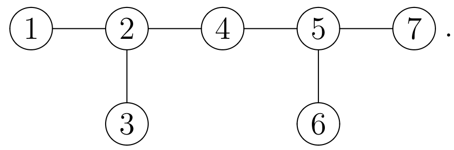
Our technique for constructing matrices with a given graph and eigenvalue constraints proceeds in two steps:

(i) Construct an "initial point"—a matrix that satisfies the eigenvalue constraints and whose graph is a subgraph of the desired graph (in terms of edge containment).

(ii) Fix the graph using the implicit function theorem, perturbing the necessary entries from zero to nonzero.

The following example illustrates the idea.

Consider the tree T,



What are the possible ordered multiplicities for a symmetric matrix with graph T? By Theorems 2.1 and 2.2, the highest multiplicity is no more than 3, and the number of distinct eigenvalues is at least 5.

In light of Theorem 2.3, if there is an eigenvalue of multiplicity 3, then the ordered multiplicities must be $1, 1, 3, 1, 1$. In fact, there is such a matrix—the adjacency matrix of T. Removing the degree 3 vertices leaves the 5-by-5 zero matrix, and the interlacing inequalities force $m(0) \geq 5 - 2 = 3$.

Does there exist an $A = (a_{ij}) \in S(T)$ with two eigenvalues $\mu \neq \nu$, each with multiplicity 2? Unlike the previous case, there appears to be no easy construction, but it is not excluded by any known condition and does, in fact, exist. Let $A[S]$ denote the principal submatrix of A lying in the rows and columns indexed by $S \subseteq \{1, \ldots, n\}$. Sufficient conditions for having

$m_A(\mu) = m_A(\nu) = 2$ are

$$a_{11} - \mu = 0, \tag{1}$$

$$a_{33} - \mu = 0, \tag{2}$$

$$\det(A[4,5,6,7] - \mu I_4) = 0, \tag{3}$$

$$\det(A[1,2,3,4] - \nu I_7) = 0, \tag{4}$$

$$a_{66} - \nu = 0, \tag{5}$$

$$a_{77} - \nu = 0, \tag{6}$$

because then we would have $m_{A(2)}(\mu) \geq 3$, $m_{A(5)}(\nu) \geq 3$, and the interlacing inequalities would give $m_A(\mu) \geq 3 - 1 = 2$, $m_A(\nu) \geq 3 - 1 = 2$. We already noted that if an eigenvalue of multiplicity 3 occurred, it would be the only multiple eigenvalue, so we actually would have $m_A(\mu) = m_A(\nu) = 2$.

Let $b \neq \mu, \nu$ be a fixed real number, and think of A as a matrix-valued function of variables $x_1, x_2, a_{12}, a_{23}, a_{24}, a_{45}, a_{56}, a_{57}$:

$$A = \begin{bmatrix} \mu & a_{12} & 0 & 0 & 0 & 0 & 0 \\ a_{12} & x_1 & a_{23} & a_{24} & 0 & 0 & 0 \\ 0 & a_{23} & \mu & 0 & 0 & 0 & 0 \\ 0 & a_{24} & 0 & b & a_{45} & 0 & 0 \\ 0 & 0 & 0 & a_{45} & x_2 & a_{56} & a_{57} \\ 0 & 0 & 0 & 0 & a_{56} & \nu & 0 \\ 0 & 0 & 0 & 0 & a_{57} & 0 & \nu \end{bmatrix}.$$

Note that if all $a_{ij} \neq 0$, then $A \in S(T)$, and also that the constraints (1, 2, 5, 6) hold for all choices of a_{ij}. Let

$$\begin{aligned} F &= (\det(A[4,5,6,7] - \mu I_4), \det(A[1,2,3,4] - \nu I_4)), \\ J &= \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \det(A[4,6,7] - \mu I_3) \\ \det(A[1,3,4] - \nu I_3) & 0 \end{bmatrix}. \end{aligned}$$

By trial and error, we find that the initial matrix $A^{(0)} = \mathrm{diag}(\mu, \nu, \mu, b, \mu, \nu, \nu)$ satisfies

$$F(A^{(0)}) = (0,0),$$

$$\det J(A^{(0)}) = \begin{vmatrix} 0 & (b-\mu)(\nu-\mu)^2 \\ (b-\nu)(\mu-\nu)^2 & 0 \end{vmatrix} \neq 0.$$

Therefore, by the implicit function theorem, we can choose $y = (a_{12}, a_{23}, a_{24}, a_{45}, a_{56}, a_{57})$ with sufficiently small nonzero entries so that equations (1 – 6) are satisfied for some pair (x_1, x_2) and $A((x_1, x_2), y) \in S(T)$. (Note that the determinant is a polynomial and thus continuously differentiable.)

Thus, the answer is yes: there does exist a symmetric matrix with graph T and two eigenvalues, each of multiplicity 2, by applying the implicit function theorem in this way.

We will see later that the unordered multiplicities for T are precisely the sequences that are majorized by $p(T), 1, 1, \ldots, 1 = 3, 1, 1, 1, 1$. Recall that

$p(T)$ is the path cover number of T, and that a non-increasing sequence $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_k$ is *majorized* by $\beta = \beta_1, \beta_2, \ldots, \beta_k$ if $\sum_{i=1}^{l} \alpha_i \leq \sum_{i=1}^{l} \beta_i$ for $l = 1, \ldots, k-1$ and $\sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{k} \beta_k$. (Pad α or β with zeros to make both the same length, if necessary.)

4 Main theorem

We can enforce a number of eigenvalue constraints on an n-by-n symmetric matrix $A = (a_{ij})$ by requiring that $\det(A[S] - \lambda I) = 0$ for various choices of $S \subseteq \{1, \ldots, n\}$ and $\lambda \in \mathbb{R}$. For convenience, if $f(A) = \det(A[S] - \lambda I)$, in which f is viewed as a function of the "variables" in A, we will abuse notation and write $f(A[R]) = \det(A[S \cap R] - \lambda I)$. (We follow the convention that the determinant of an empty matrix is 1, so that, in particular, $f(A[R]) = 1$ if $S \cap R = \varnothing$.)

Given a tree T on n vertices and a *vector of determinant conditions* $F = (f_k)$, $f_k(A) = \det(A[S_k] - \lambda_k I)$, we wish to show the existence of a symmetric matrix A with graph T that satisfies $F(A) = 0$.

To do this, we will construct an initial n-by-n matrix $A^{(0)} = (a_{ij}^{(0)})$ for which $F(A^{(0)}) = 0$ and the graph of $A^{(0)}$ is a subgraph of T (in terms of edge containment). Then we will perturb some entries of $A^{(0)}$ as we see fit, and the implicit function theorem will perturb the remaining entries in order to maintain the eigenvalue constraints specified by F. We will designate the entries to be "manually" perturbed as *manual entries*, and the entries to be “implicitly” perturbed as *implicit entries*. In the example of section 3, the manual entries were a_{12}, a_{23}, a_{24}, a_{45}, a_{56}, and a_{57}, and the implicit entries were x_1 and x_2. Because of the Jacobian requirement in the implicit function theorem, if F is a vector of length r, then precisely r of the independent entries must be designated as implicit. (Note that because of the prevailing symmetry requirement, a symmetrically placed pair of off-diagonal entries is not independent.)

Each implicit entry is simply an unknown entry a_{ij}, $1 \le i \le j \le n$. However, we must choose the implicit entries $a_{i_1,j_1}, \ldots, a_{i_r,j_r}$ so that the Jacobian

$$\left| \frac{\partial F}{\partial a_{i_1,j_1} \cdots \partial a_{i_r,j_r}} \right|$$

is nonzero. The following two lemmas will be helpful.

Lemma 4.1. *Let T and $F = (f_k)_{k=1,\ldots,r}$ be defined as above, with r implicit entries identified. Suppose that a symmetric matrix $A^{(0)}$, whose graph is a subgraph of T, is the direct sum of irreducible matrices $A_1^{(0)}, A_2^{(0)}, \ldots, A_p^{(0)}$. Let $J(A^{(0)})$ be the Jacobian matrix of F with respect to the implicit entries evaluated at $A^{(0)}$, and suppose*

(i) Every off-diagonal implicit entry in $A^{(0)}$ has a nonzero value.

(ii) For every $k = 1, \ldots, r$, $f_k(A_l^{(0)}) = 0$ for precisely one $l \in \{1, \ldots, p\}$.

(iii) For every $l = 1, \ldots, p$, the columns of $J(A^{(0)})$ associated with the implicit entries of $A_l^{(0)}$ are linearly independent.

Then $J(A^{(0)})$ is nonsingular.

According to the lemma, we can take advantage of the reducibility of a matrix to decide if a Jacobian of determinant conditions is nonzero.

Proof. First, we find formulas for the entries of the Jacobian matrix. Recall that $f_k(A) = \det(A[S_k] - \lambda_k I)$. Suppose that a_{ii} is a diagonal implicit entry. If $i \in S_k$, then $f_k(A)$ has the form $(a_{ii} - \lambda_k)\det(A[S_k \smallsetminus i] - \lambda_k I) + \cdots$, with the remaining terms independent of a_{ii}, and if $i \notin S_k$, then $f_k(A)$ is completely independent of a_{ii}. Therefore,

$$\frac{\partial f_k}{\partial a_{ii}} = \begin{cases} \det(A[S_k \smallsetminus i] - \lambda_k I) & \text{if } i \in S_k \\ 0 & \text{otherwise} \end{cases}.$$

In particular, at the initial matrix,

$$\frac{\partial f_k}{\partial a_{ii}}(A^{(0)}) = \begin{cases} \prod_{l=1}^{p} f_k(A_l^{(0)}[S_k \smallsetminus i]) & \text{if } i \in S_k \\ 0 & \text{otherwise} \end{cases}, \tag{7}$$

in which the rows and columns of $A_l^{(0)}$ retain their indices from $A^{(0)}$. Now consider any off-diagonal implicit entry a_{ij}. Because every edge of a tree is a cut-edge, any matrix $A \in S(T)$ is permutation-equivalent to a matrix of the form

$$\left[\begin{array}{c|c} \text{B} & \\ & a_{ij} \\ \hline a_{ij} & \\ & \text{C} \end{array}\right].$$

By a simple counting argument, one can show that any nonzero term in $\sum_{\text{distinct } \pi_i} a_{\pi_i,i} = \det(A[S_k] - \lambda_k I)$ including the factor a_{ij} in the top-right block must also include the factor a_{ij} from the bottom-left block. Therefore, if $i, j \in S_k$, then

$$f_k(A) = \det(B - \lambda_k I)\det(C - \lambda_k I) - a_{ij}^2 \det(A[S_k \setminus \{i,j\}] - \lambda_k I),$$

and if i or j is not in S_k, then $f_k(A)$ is completely independent of a_{ij}. Hence,

$$\frac{\partial f_k}{\partial a_{ij}} = \begin{cases} -2a_{ij}\det(A[S_k \setminus \{i,j\}] - \lambda_k I) & \text{if } i,j \in S_k \\ 0 & \text{otherwise} \end{cases}.$$

In particular, when evaluated at the initial matrix,

$$\frac{\partial f_k}{\partial a_{ij}}(A^{(0)}) = \begin{cases} -2a_{ij}^{(0)} \prod_{l=1}^{p} f_k(A_l^{(0)}[S_k \setminus \{i,j\}]) & \text{if } i,j \in S_k \\ 0 & \text{otherwise} \end{cases}. \tag{8}$$

We have now found formulas for all entries of the Jacobian matrix evaluated at $A^{(0)}$.

Suppose an implicit entry a_{ij} lies in $A_l^{(0)}$. If $f_k(A_l^{(0)}) \neq 0$, then by (ii), $f_k(A_z^{(0)}) = 0$ for some $z \neq l$, and so (7–8) imply

$$\frac{\partial f_k}{\partial a_{ij}}(A^{(0)}) = 0.$$

Thus, every nonzero entry in the kth row of $J(A^{(0)})$ occurs in a column which corresponds to the unique direct summand $A_l^{(0)}$ for which $f_k(A_l^{(0)}) = 0$. By (iii) and a counting argument, $J(A^{(0)})$ is permutation equivalent to a block diagonal matrix whose blocks are square and nonsingular. □

As a special case of Lemma 4.1, evaluation of a Jacobian of determinant conditions, at a diagonal matrix, is straightforward.

Lemma 4.2. *Let $F = (f_k)$ be a vector of r determinant conditions, and let $A^{(0)}$ be a diagonal matrix. Suppose that for every $k = 1, \ldots, r$, $f_k(A^{(0)}[l]) = 0$ for precisely one $l \in \{1, \ldots, n\}$. Take a_{ll} to be an implicit entry if and only if $f_k(A^{(0)}[l]) = 0$ for some k. If there are then r implicit entries, the Jacobian of F with respect to the implicit entries evaluated at $A^{(0)}$ is nonzero.*

Proof. We apply Lemma 4.1. In this application, $p = n$ and every direct summand $A_1^{(0)}, \ldots, A_n^{(0)}$ is a scalar. The key point to prove is that the columns of $J(A^{(0)})$ associated with the implicit entries in $A_l^{(0)}$ are nonsingular. Well, since there can be at most one implicit entry in the 1-by-1 $A_l^{(0)}$, it suffices to check that every column of $J(A^{(0)})$ is nonzero. If a_{ll} is implicit, then $f_k(A^{(0)}[l]) = 0$ for some k. Consider $\frac{\partial f_k}{\partial a_{ll}}$. This derivative equals $\prod_{i \in S_k,\, i \neq l}(a_{ii} - \lambda_k)$. When evaluating at the initial matrix, we substitute $a_{ii}^{(0)}$ for a_{ii}. By assumption, $f_k(A^{(0)}[i]) = 0$ only if $i = l$, i.e., the only value of i for which $a_{ii}^{(0)} = \lambda_k$ is l, so $\frac{\partial f_k}{\partial a_{ll}}(A^{(0)}) \neq 0$. Lemma 4.1 applies. $\square$

We will often use Lemma 4.1 or Lemma 4.2 to check the Jacobian more easily.

Definition 4.3. A sequence of integers $\beta = (\beta_1, \ldots, \beta_l)$ is a *refinement* of a sequence of integers $\alpha = (\alpha_1, \ldots, \alpha_k)$ if β can be obtained from α by replacing each α_i by an ordered partition of α_i. For example, 112121 is a refinement of 11321.

Theorem 4.4. *Let G be a graph, let $F = (f_k)$ be a vector of r determinant conditions, and designate r entries as implicit entries. Suppose a symmetric matrix $A^{(0)}$ is the direct sum of irreducible matrices $A_1, A_2, \ldots, A_p$, and*

(i) $F(A^{(0)}) = 0$.

(ii) $G(A^{(0)})$ *is a subgraph of* G.

(iii) If $a_{ij}^{(0)} = 0$, $i \neq j$, *then* (i, j) *is* not *an implicit entry.*

(iv) $\det J(A^{(0)})$, *the Jacobian of F with respect to the implict entries, evaluated at $A^{(0)}$, is nonzero.*

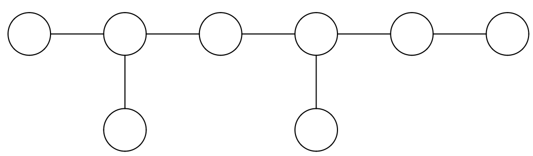
Then there exists a symmetric matrix A with graph G such that $F(A) = 0$ and the ordered multiplicities of A are a refinement of the ordered multiplicities of $A^{(0)}$.

The usefulness of the theorem depends on the particular choice of F. We will construct F so that certain principal submatrices of A are forced to have particular eigenvalues. Then the interlacing inequalities will give lower bounds on the multiplicities of these eigenvalues in the entire matrix A.

Proof. At the initial matrix $A^{(0)}$, the Jacobian is nonzero. Therefore, we can apply the implicit function theorem. If i and j are adjacent in G and $a_{ij}^{(0)} = 0$, then (i, j) is not an implicit entry, so we may perturb each such entry from zero to nonzero. By the implicit function theorem, if each perturbation is sufficiently small, then there exists a choice for the implicit entries so that the corresponding matrix A has $F(A) = 0$. Sufficiently small perturbations also guarantee that A has graph G and that no two distinct eigenvalues coalesce (since the eigenvalues are continuous functions of the entries). Therefore, the ordered multiplicities of A are a refinement of the ordered multiplicities of $A^{(0)}$. □

5 Unordered multiplicities for a class of graphs

A *vine* is a binary tree in which every degree 3 vertex is adjacent to at least one degree 1 vertex, and no two degree 3 vertices are adjacent. Every vine has a diameter from which every other vertex hangs, e.g.



Note that the path cover number (see Theorem 2.1) of a vine is one more than the number of degree 3 vertices.

Before stating our main result for this section—a description of the possible unordered multiplicity lists for a vine—we must review some facts that were instrumental in proving Theorems 2.1 and 2.3. As mentioned earlier, any symmetric matrix A, real number λ, and index i satisfy $|m_A(\lambda) - m_{A(i)}(\lambda)| \leq 1$. For historical reasons [P], if $m_A(\lambda) - m_{A(i)}(\lambda) = -1$, we call i a *Parter vertex* (for A and λ). In [JLDSa], it was shown that if the graph of A is a tree and if $\lambda \in \sigma(A) \cap \sigma(A(j))$ for some j, then A must have a Parter vertex for λ. This encompassed earlier work by Parter and then Wiener.

If the graph of A is a tree, then the removal of any vertex i leaves a number of *branches at* i, i.e. connected components of the vertex-deleted graph, each a tree. There is a stronger statement in [JLDSa], specifically, if $m_A(\lambda) \geq 2$, then A must have a *strong Parter vertex*. A strong Parter vertex for λ is a Parter vertex i of degree ≥ 3 for which λ is an eigenvalue of at least three branches (direct summands) of $A(i)$. Strong Parter vertices are key to the proof of the next lemma.

Lemma 5.1. *If the graph of a symmetric matrix $A = (a_{ij})$ is a vine T and $\lambda_1, \ldots, \lambda_l$ are the distinct eigenvalues of A, then T has at least $\sum_{i=1}^{l}(m_A(\lambda_i) - 1)$ degree 3 vertices.*

Proof. Suppose $m_A(\lambda) \geq 2$. Then there exists a strong Parter vertex i for λ. If the multiplicity of λ in any branch at i is at least 2, then we can repeat the process, finding more strong Parter vertices. In the end, $m_A(\lambda) - 1$ strong Parter vertices can be found sequentially. It remains to show that no two eigenvalues can share a Parter vertex, so that no vertex is doubly counted. For this, observe that if i is Parter for λ, then i is adjacent to at least one degree 1 vertex, and that $a_{jj} = \lambda$ for every adjacent degree 1 vertex j. $\square$

Lemma 5.2. *Suppose that T is a vine on n vertices and that*

$$m_1, \ldots, m_l, 1, \ldots, 1$$

is an unordered multiplicity list that partitions n, with $m_i \geq 2$, $i = 1, \ldots, l$. If $\sum_{i=1}^{l}(m_i - 1)$ is no more than the number of degree 3 vertices in T, then there exists a symmetric matrix $A \in S(T)$ with the given multiplicities.

Proof. Choose any numerical values $\lambda_1, \ldots, \lambda_l$. Identify a diameter of T, placing one end on the "left" and the other on the "right." Every vertex is thus "on the diameter" or a "hanging pendant." The leftmost $m_1 - 1$ degree 3 vertices will eventually be Parter for λ_1, the next $m_2 - 1$ will be Parter for λ_2, and so on. For convenience, we will immediately refer to these as Parter vertices, even though we have not yet constructed a matrix. The set of Parter vertices for λ_i will be denoted V_i.

The vector of determinant conditions $F(A)$ has $\sum_{i=1}^{l}(2m_i - 1)$ entries. Each entry is of the form $\det(A[S] - \lambda I)$, in which λ is one of $\lambda_1, \ldots, \lambda_l$ and S identifies one of the branches obtained from deletion of the Parter vertices for λ.

The initial condition is a diagonal matrix $A^{(0)} = (a_{ij}^{(0)})$. In constructing this matrix, it is helpful to label certain vertices of T. For $i = 1, \ldots, l$, find the $m_i - 1$ Parter vertices for λ_i. For each, label the neighbor on the diameter immediately to the right and also the adjacent hanging pendant with the label λ_i. This produces $2(m_i - 1)$ vertices labeled λ_i. Finally, label the very leftmost vertex on the diameter with λ_1, and, for $i = 2, \ldots, l$, label the rightmost Parter vertex for λ_{i-1} with the label λ_i. In this way, no vertex is labeled twice, and the deletion of V_i leaves $2m_i - 1$ branches, each containing exactly one vertex labeled λ_i. Now, construct the diagonal matrix $A^{(0)}$ by setting $a_{jj} = \lambda_i$ if vertex j is labeled with λ_i, and by ensuring that all other entries are not equal to $\lambda_1, \ldots, \lambda_l$ or to each other.

The implicit entries are precisely the entries corresponding to labeled vertices. The manual entries are the off-diagonal entries that are allowed to be nonzero by the graph.

Because $F(A^{(0)}) = 0$, all implicit entries are diagonal entries, and the Jacobian is nonzero at $A^{(0)}$ (by Lemma 4.2), Theorem 4.4 shows that there exists a matrix $A = (a_{ij})$ with graph T such that $F(A) = 0$. This implies that λ_i is an eigenvalue of each of the $2m_i - 1$ direct summands of $A(V_i)$. By the interlacing inequalities, $m_A(\lambda_i) \geq (2m_i - 1) - \#V_i = m_i$.

The proof will be complete after placing upper bounds on the multiplicities. First consider λ_i. If $m_A(\lambda_i)$ were greater than m_i, then λ_i would be a multiple eigenvalue of one of the direct summands of $A(V_i)$. However, the multiplicity of λ_i in each direct summand of $A^{(0)}(V_i)$ is at most 1, so by choosing a small enough perturbation, λ_i can be guaranteed not to be a multiple eigenvalue of any direct summand of $A(V_i)$. Next, consider the remaning eigenvalues, that are intended to have multiplicity 1. To see that they must, in fact, be singletons, it suffices to show that no eigenvalue other than $\lambda_1, \ldots, \lambda_l$ has a strong Parter vertex. For a vine, no two eigenvalues may share a Parter vertex, so consider a degree 3 vertex v that is not Parter for any λ_i. v is adjacent to a hanging pendant u, whose corresponding entry is neither implicit nor manual, i.e., it remains equal to $a_{uu}^{(0)}$ even after applying the implicit function theorem. By choosing the perturbation to be sufficiently small, A can be guaranteed not to have a_{uu} as an eigenvalue of any other direct summand of $A(v)$. This guarantees that v is not a Parter vertex for any eigenvalue. □

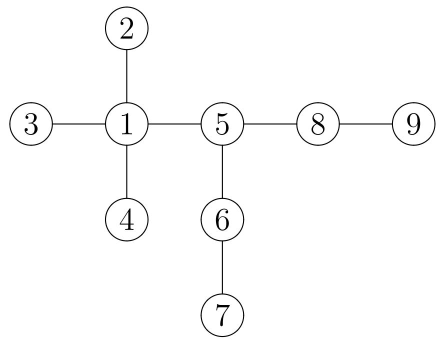
The main theorem of this section is a direct consequence of the two lemmas.

Theorem 5.3. *The possible unordered multiplicities for a vine T on n vertices are the sequences of positive integers that are majorized by* $p(T), 1, 1, \ldots, 1$.

Call a tree a *generalized vine* if every vertex of degree $k \geq 3$ is adjacent to at least $k - 2$ pendant vertices and no vertex of degree at least three is adjacent to another. Like a vine, a generalized vine has a diameter from which every other vertex hangs and the degree at least three vertices are separated, but a generalized vine is not necessarily binary. It is worth noting that for a generalized vine T, when the multiplicity $p(T)$ is attained, all other multiplicities are 1. This is a simple consequence of the fact that $p(T)$ is the number of pendant vertices less 1 for a generalized vine and that the number of distinct eigenvalues is always at least the diameter (measured in vertices) [LDJ], which is the total number of vertices, less the pendant vertices, plus 2 for a generalized vine.

6 Implicitly constructing eigenvalues

Our implicit construction is useful for trees besides vines. Let T be the tree



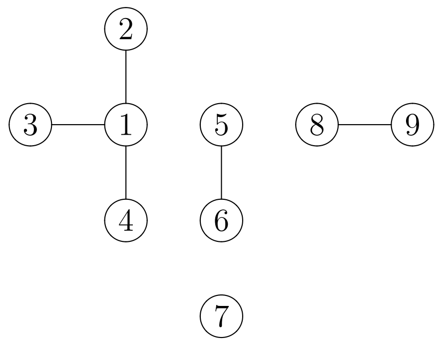
We will construct a symmetric matrix with graph T and spectrum -2, -1, -1, -1, 0, 1, 1, 2, 3. Note that all 9 eigenvalues of the 9-by-9 matrix are specified in advance.

Let

$$F = \begin{pmatrix} \det(A - (-2)I_9) \\ \det(A[2] - (-1)I_1) \\ \det(A[3] - (-1)I_1) \\ \det(A[4] - (-1)I_1) \\ \det(A[1,2,3,4] - 1I_4) \\ \det(A - 0I_9) \\ \det(A - 2I_9) \\ \det(A[6,7] - 1I_2) \\ \det(A[5,6,7,8,9] - (-1)I_5) \\ \det(A[8,9] - 1I_2) \\ \det(A - 3I_9) \end{pmatrix}.$$

Any 9-by-9 matrix in the kernel of F must have spectrum $-2, -1, -1, -1, 0, 1, 1, 2, 3$.

We can construct a symmetric matrix $A^{(0)} = (a_{ij}^{(0)})$ with $F(A^{(0)}) = 0$ and graph

$T' =$ 

without too much trouble. The following matrix works:

$$A^{(0)} = \begin{bmatrix} 0 & \frac{1}{2}\sqrt{6} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\sqrt{6} & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5}{2} & \frac{1}{2}\sqrt{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{3} & \frac{3}{2} \end{bmatrix}$$

Allowing entries $(1,5)$, $(5,8)$, and $(6,7)$ to be zero made the construction of $A^{(0)}$ straightforward.

Now we need to perturb entries $(1,5)$, $(5,8)$, and $(6,7)$, along with their symmetrically placed mates, in order to find a matrix $A^{(1)}$ close to $A^{(0)}$ which has graph T and $F(A^{(1)}) = 0$. Take entries $(1,1)$, $(1,2)$, $(2,2)$, $(3,3)$, $(4,4)$,

$(5,5)$, $(5,6)$, $(6,6)$, $(7,7)$, $(8,8)$, $(9,9)$ to be implicit entries. All other entries are manual entries.

Consider Lemma 4.1. Hypothesis (i) of the lemma is clearly satisfied. Checking hypothesis (ii) requires more work. The first entry of F is $\det(A-(-2)I_9)$. Therefore, we must verify that -2 is an eigenvalue of exactly one of the direct summands of $A^{(0)}$. The second entry is $\det(A[2]-(-1)I_1)$. Therefore, the $(2,2)$-entry of $A^{(0)}$ must be -1. The next few entries are similar, but the entry $\det(A[6,7]-1I_2)$ is worth addressing. We must check that 1 is an eigenvalue of exactly one direct summand of $A[6,7]=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. We omit the remaining details. For hypothesis (iii), check that the sets of columns $\{1,2,3,4,5\}$, $\{6,7,8\}$, $\{9\}$, and $\{10,11\}$ of the Jacobian matrix of F with respect to the implicit entries, evaluated at $A^{(0)}$, are linearly independent. There is some subtlety in constructing $A^{(0)}$ so that these columns are linearly independent. In particular, the last two columns of the Jacobian matrix evaluated at $A^{(0)}$ are zero in the first nine rows (see the discussion regarding the essentially block diagonal form of the Jacobian matrix in the proof of Lemma

4.1), so we need the following inequality to hold when evaluated at $A^{(0)}$:

$$\det \begin{bmatrix} \frac{\partial}{\partial a_{88}} \det(A[8,9] - 1I_2) & \frac{\partial}{\partial a_{99}} \det(A[8,9] - 1I_2) \\ \frac{\partial}{\partial a_{88}} \det(A - 3I_9) & \frac{\partial}{\partial a_{99}} \det(A - 3I_9) \end{bmatrix}$$

$$= \det \begin{bmatrix} a_{99} - 1 & a_{88} - 1 \\ (a_{99} - 3)\det(A(8,9) - 3I_7) & (a_{88} - 3)\det(A(8,9) - 3I_7) - a_{58}^2 \det(A(5,8,9) - 3I_6) \end{bmatrix}$$

$$\neq 0.$$

Since the initial point is 0 in the (5, 8) entry, we need

$$\det \begin{bmatrix} a_{99}^{(0)} - 1 & a_{88}^{(0)} - 1 \\ (a_{99}^{(0)} - 3)\det(A^{(0)}(8,9) - 3I_7) & (a_{88}^{(0)} - 3)\det(A^{(0)}(8,9) - 3I_7) \end{bmatrix} \neq 0,$$

i.e. $a_{88}^{(0)} \neq a_{99}^{(0)}$.

Lemma 4.1 is useful because it guarantees that the constraints imposed by a nonzero Jacobian affect the direct summands of $A^{(0)}$ independently. For example, above we needed $a_{88}^{(0)} \neq a_{99}^{(0)}$, but the constraint did not involve any entries of $A^{(0)}[1,2,3,4,5,6,7]$.

Now we can apply Theorem 4.4. We constructed $A^{(0)}$ so that hypotheses (i) and (ii) of this theorem hold. Our only off-diagonal implicit entries, $(1,2)$ and $(5,6)$, are nonzero entries, so hypothesis (iii) holds. Finally, we just checked that the Jacobian is nonzero. Therefore, there exists a symmetric matrix $A^{(1)}$ with graph T such that $F(A^{(1)}) = 0$. We already noted that this implies $\sigma(A^{(1)}) = -2, -1, -1, -1, 0, 1, 1, 2, 3$.

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