Chapter 9

Danger from uniform grids of high degree

We have seen that higher-degree polynomials often provide better fits than lower-degree polynomials. Why bother with *piecewise*-polynomial interpolation at all? Why not just fit a function with a single high-degree polynomial? Eventually we will do this, but there is danger ahead.

9.1 ▪ The Runge phenomenon

One danger of high-degree polynomial interpolation is exemplified by a function called *Runge's function*.

Example 9.1. Runge's function is

$$f(x) = \frac{1}{1 + 25x^2}.$$

Interpolate this function on a degree-m uniform grid over $-1 \le x \le 1$. Investigate the interpolation error as m increases.

Solution. In Figure 9.1, Runge's function is accompanied by polynomial interpolants of degrees $m = 5$, 10, and 20.

```
>> f = @(x) 1/(1+25*x^2);
>> a = -1;
>> b = 1;
>> m = [5 10 20];
>> ax = newfig(1,3);
>> for k = 1:length(m)
     subplot(ax(k));
     ps = sampleuni(f,[a b],m(k),1);
     p = interpuni(ps,[a b]);
     plotfun(f,[a b],'displayname','f');
     plotsample(griduni([a b],m(k),1),ps);
     plotfun(p,[a b],'displayname','p');
     ylim([-2 2]);
     xlabel('x');
   end
```

As the degree m increases, the quality of the interpolation worsens outside of a central interval, contrary to what we might expect. ∎

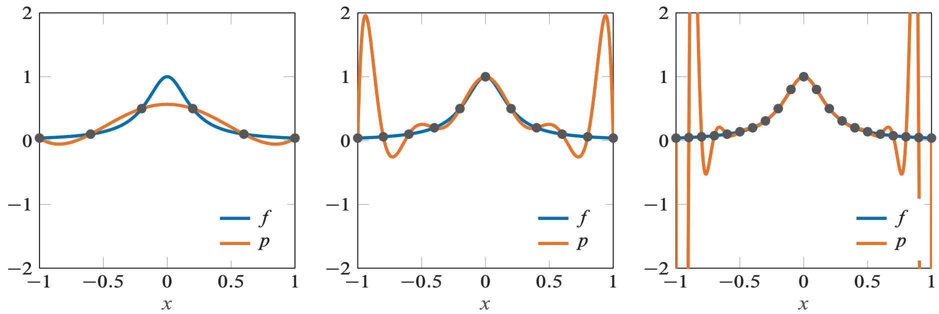

Figure 9.1. *Polynomial interpolants for Runge's function using uniform grids of degrees $m = 5$ (left), $m = 10$ (middle), and $m = 20$ (right).*

The example illustrates the *Runge phenomenon*, characterized by wild oscillations at the ends of an interpolant that become more severe as the degree is increased. More work leads to poorer results. This is unacceptable.

Why does the Runge phenomenon occur? It is important to realize that Runge's function is not as benign as it first seems. To see this requires a peek into the complex plane and a hunt for *singularities*, points where a function is undefined or not complex differentiable. (See Appendix B for a brief primer on complex functions.)

Example 9.2. Plot the real and imaginary parts of Runge's function $f(z) = 1/(1 + 25z^2)$ over the domain $z = x + y\mathrm{i}$ with $-1 \leq x \leq 1$, $-0.4 \leq y \leq 0.4$. Identify any singularities.

Solution. The real and imaginary parts are graphed in Figure 9.2 using new routines `plotrealpart` and `plotimagpart`.

```
>> f = @(z) 1/(1+25*z^2);
>> ax = newfig(1,2);
>> legend hide;
>> plotrealpart(f,[-1 1 -0.4 0.4 -2 2]);
>> view(-25,30);
>> xlabel('x'); ylabel('y');
>> subplot(ax(2));
>> legend hide;
>> plotimagpart(f,[-1 1 -0.4 0.4 -2 2]);
>> view(-25,30);
>> xlabel('x'); ylabel('y');
```

In the above code, the vector `[-1 1 -0.4 0.4 -2 2]` sets the limits $[-1, 1]$ on the x-axis, $[-0.4, 0.4]$ on the y-axis, and $[-2, 2]$ on the vertical axis.

Note that the hump shape of Runge's function on the real line, familiar from Figure 9.1, is visible along the slice $y = 0$. This portion of the graph is drawn with a heavy pen for emphasis.

```
>> subplot(ax(1));
>> plotfun3(@(x) x,@(x) 0,@(x) real(f(x)),[-1 1],'k','linewidth',3);
>> subplot(ax(2));
>> plotfun3(@(x) x,@(x) 0,@(x) imag(f(x)),[-1 1],'k','linewidth',3);
```

Runge's function has exactly two singularities. Specifically, at $z = \pm(1/5)\mathrm{i}$, the function has the form $f(\pm(1/5)\mathrm{i}) = 1/0$, and in a neighborhood of either point the function is unbounded. The plots show infinite discontinuities called *poles* at these points. ∎

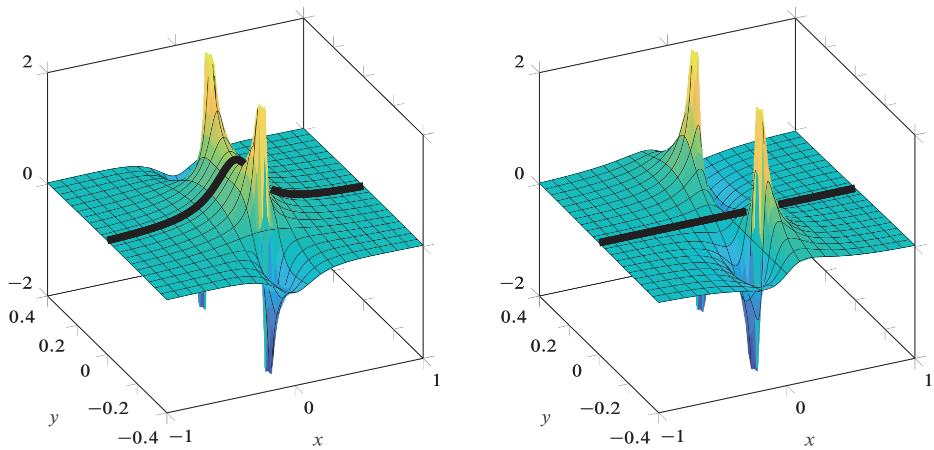

Figure 9.2. *The real (left) and imaginary (right) parts of Runge's function* $f(z) = 1/(1+25z^2)$ *in the complex plane.*

Although Runge's function looks well behaved on the real line, sea monsters lurk nearby in the complex plane. The function is undefined at $z = \pm(1/5)\mathrm{i}$, and its absolute value diverges to infinity as either of these points is approached. Perhaps a singularity near the interpolation interval can cause problems for polynomial interpolation, even if that singularity lies off the real line.

Indeed, the Runge phenomenon occurs when a singularity is too close to the interpolation interval (and a uniform grid of high degree is used). Exactly how close is too close is determined by a particular curve in the complex plane that we call the *Runge curve*. When there is a singularity enclosed by the Runge curve, the Runge phenomenon may occur. Although we do not have a simple formula for this curve, it can be drawn with the routine `rungecurve` demonstrated in the following example. The Runge curve does not depend on the degree m of the grid, and the curve for $[a,b]$ is a simple translation and rescaling of the one for $[-1,1]$, so once you've seen a single Runge curve, you've essentially seen them all.

Example 9.3. Plot the Runge curve for the interval $[-1,1]$ and the locations of the singularities of Runge's function. Comment on the implications for interpolation with a uniform grid of high degree.

Solution. The Runge curve is plotted in Figure 9.3.

```
>> a = -1;
>> b = 1;
>> newfig; legend hide;
>> rungecurve([a b]);
>> axis([-2 2 -2 2]);
>> axis square; grid on;
>> xlabel('real'); ylabel('imag');
```

Runge's function $f(z) = 1/(1+25z^2)$ has singularities at $\pm(1/5)\mathrm{i}$. These are marked by $\times$'s in the plot.

```
>> plot([0 0],[-1/5 1/5],'x');
```

Because the singularities lie within the Runge curve, interpolation of $f(x) = 1/(1+25x^2)$ is susceptible to the Runge phenomenon. As we saw in Example 9.1, the Runge phenomenon does indeed occur. ∎

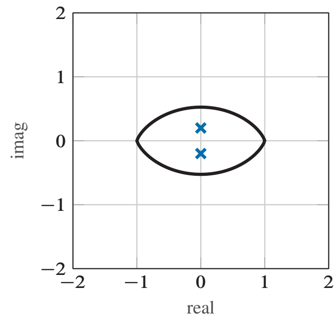

Figure 9.3. *The Runge curve for* $[-1, 1]$ *and the singularities of Runge's function.*

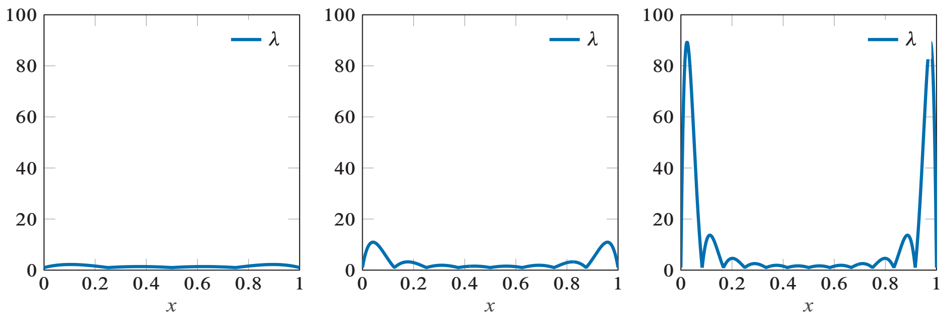

Figure 9.4. *Lebesgue function for a uniform grid of degree* m*:* $m = 4$ *(left),* $m = 8$ *(middle), and* $m = 12$ *(right).*

Note that the Runge curve for $[a, b]$ intersects the points a and b on the real line.

9.2 ▪ The Lebesgue constant for uniform grids

The Runge phenomenon isn't the end of the story. Just as damaging is the Lebesgue constant for a uniform grid of high degree. Recall that the Lebesgue constant was introduced in advance of Theorem 6.11 to quantify the effect of sampling errors on an interpolant. The conclusion was that sampling errors of size δ perturb the entire interpolant proportionally, up to $\Lambda\delta$ in the infinity norm, and the constant of proportionality Λ is the Lebesgue constant. Unfortunately, the Lebesgue constant for a high-degree uniform grid is huge, as proved below, so even tiny errors at the interpolation nodes can be blown into huge errors between the nodes.

The Lebesgue constant Λ for a grid is

$$\Lambda = \max_{a \le x \le b} \lambda(x),$$

in which $\lambda(x)$ is the Lebesgue function defined in (6.9). The Lebesgue functions for three uniform grids are plotted in Figure 9.4. The Lebesgue functions, and therefore the Lebesgue constants, appear to grow very quickly with m, a fact established by the following theorem.

Theorem 9.4. *The Lebesgue constant* Λ_m *for a uniform grid of degree* $m \ge 1$ *satisfies*

$$\Lambda_m \ge \frac{2^m}{4m^2}. \tag{9.1}$$

Proof. The Lebesgue constant for a uniform grid of degree m is independent of the interval on which the grid is placed (Exercise 18 in Chapter 8). For convenience, place the grid on $[a,b] = [0,m]$, so that the nodes are $x_i = i$, $i = 0,\dots,m$.

The ith Lagrange basis polynomial is, when $x \neq i$,

$$l_i(x) = \frac{\prod_{k=0}^{m}(x - x_k)}{x - x_i} = \frac{\prod_{k=0}^{m}(x-k)}{x-i},$$

and moreover,

$$\begin{aligned} l_i(x_i) = l_i(i) &= \prod_{k=0}^{i-1}(i-k) \prod_{k=i+1}^{m}(i-k) \\ &= i(i-1)\cdots(1) \times (-1)(-2)\cdots(-(m-i)) = i! \times (-1)^{m-i}(m-i)!. \end{aligned}$$

The Lebesgue constant is the maximum of the Lebesgue function on $[a,b] = [0,m]$. Figure 9.4 suggests an investigation near one of the endpoints. It will prove sufficient to concentrate on $x = 1/2$, halfway between the first two nodes. We find

$$l_i(1/2) = \frac{\left(\frac{1}{2}-0\right)\left(\frac{1}{2}-1\right)\prod_{k=2}^{m}\left(\frac{1}{2}-k\right)}{\frac{1}{2}-i} = (-1)^{m-1}\frac{\prod_{k=2}^{m}\left(k-\frac{1}{2}\right)}{4\left(i-\frac{1}{2}\right)}.$$

Therefore,

$$|l_i(1/2)| \geq \frac{\prod_{k=2}^{m}(k-1)}{4m} = \frac{(m-1)!}{4m} = \frac{m!}{4m^2}.$$

The Lebesgue function at $x = 1/2$ is bounded as follows:

$$\lambda(1/2) = \sum_{i=0}^{m}\left|\frac{l_i(1/2)}{l_i(x_i)}\right| \geq \sum_{i=0}^{m}\frac{m!/(4m^2)}{i!(m-i)!} = \frac{1}{4m^2}\sum_{i=0}^{m}\binom{m}{i},$$

in which $\binom{m}{i} = m!/(i!(m-i)!)$ is a binomial coefficient. The above sum of binomial coefficients equals 2^m. (The binomial coefficient $\binom{m}{i}$ counts the number of subsets of $\{1,\dots,m\}$ of size i. The sum therefore counts the number of subsets of any size, which equals 2^m.) Therefore,

$$\lambda(1/2) \geq \frac{2^m}{4m^2}.$$

The lower bound on $\lambda(1/2)$ provides a lower bound on the Lebesgue constant:

$$\Lambda_m = \max_{0 \leq x \leq m} \lambda(x) \geq \lambda(1/2) \geq \frac{2^m}{4m^2}. \qquad \square$$

The inequality $\Lambda_m \geq 2^m/(4m^2)$ shows that the Lebesgue constant grows very quickly—exponentially—with the degree of a uniform grid. For example, when $m = 64$, the Lebesgue constant is greater than 10^{15}.

Example 9.5. Interpolate $f(x) = e^{-x^2}$, $-6 \leq x \leq 6$, with uniform grids of increasing degree. What happens?

Solution. Three interpolants are computed and graphed in Figure 9.5.

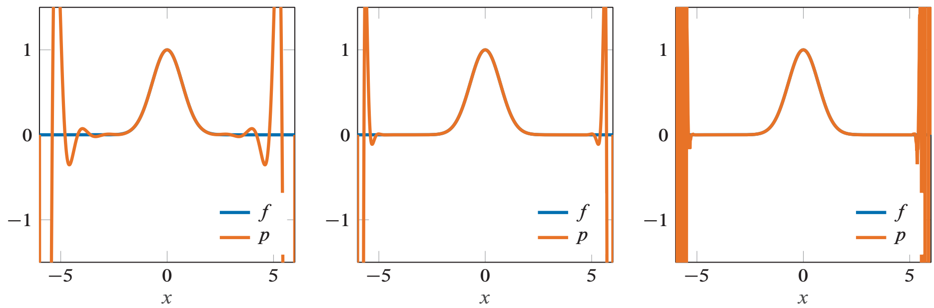

Figure 9.5. *Polynomial interpolant for e^{-x^2} on a uniform grid of degree m: $m = 20$ (left), $m = 40$ (middle), and $m = 80$ (right).*

```
>> f = @(x) exp(-x^2);
>> a = -6;
>> b = 6;
>> m = [20 40 80];
>> ax = newfig(1,3);
>> for k = 1:length(m)
     subplot(ax(k));
     ps = sampleuni(f,[a b],m(k),1);
     p = interpuni(ps,[a b]);
     plotfun(f,[a b],'displayname','f');
     plotfun(p,[a b],'displayname','p');
     xlabel('x');
     ylim([-1.5 1.5]);
   end
```

The graphed interpolants are highly inaccurate, and there is little hope that the accuracy would improve with higher degrees because the Lebesgue constant would only grow larger. ∎

Note that the behavior in the previous example is not the Runge phenomenon. The Runge phenomenon occurs when a smooth interpolant tries to fit a singularity. When this happens, the polynomial interpolants diverge in theory, before roundoff error ever comes into play. In contrast, the function e^{-x^2} of the previous example is analytic (smooth, with no singularities) over the entire complex plane. In theory, the interpolants should converge to the original function, but in practice the sensitivity to tiny sampling errors quantified by the Lebesgue constant causes the numerical computations to become inaccurate.

9.3 ▪ Conclusion

The lesson of this chapter is that polynomial interpolation performs poorly on a uniform grid of high degree. This motivates the study of nonuniform grids beginning in the next chapter.

Notes

Runge described the phenomenon named after him in 1901 [64], using the function $1/(1+x^2)$ on $[-5,5]$. Méray predicted the Runge phenomenon as early as 1884 [54] and considered Runge's function in 1896 [55]. The curve that we call the Runge curve is one of a family of level curves that Runge called *U-curves* in his original work. Epperson provides a summary of Runge's work as well as related original results [29]. The claim after Example 9.5 that the interpolants converge to $f(x) = e^{-x^2}$ is justified by the proposition on p. 331 of Epperson's article.

An article by Trefethen and Weideman discusses the history of the Lebesgue constant for a uniform grid [78]. The proof of Theorem 9.4 is based on a proof of a stronger result in Fornberg's book [32, pp. 171–172].

Exercises

Exercises 1–10: Interpolate the given function with uniform grids of increasing degree. Do the interpolants converge numerically to an accurate approximation? If so, provide evidence. If not, show what happens.

1. $1/x$, $1 \le x \le 5$
2. $1/(1+2x^2)$, $-1 \le x \le 1$
3. $1/(1+x^4)$, $-5 \le x \le 5$
4. $\tanh x = (e^x - e^{-x})/(e^x + e^{-x})$, $-3 \le x \le 3$
5. $\tanh x$, $-5 \le x \le 5$
6. $\operatorname{erf} x$, $-4 \le x \le 4$
7. $\operatorname{erf} x$, $-6 \le x \le 6$
8. $\tan x$, $-\pi/3 \le x \le \pi/3$
9. $\arctan x$, $-2 \le x \le 2$
10. $\arctan x$, $-6 \le x \le 6$

Exercises 11–18: Locate the singularities, if any, of the given function in the complex plane.

11. $z/(z^2 - 2z + 2)$
12. $(z+3)/(z^4+2)$
13. $\log z$
14. $\log(z^2+1)$
15. $\sqrt{z}$
16. $\sqrt{1-z^2}$
17. e^z
18. e^{z^2}

Exercises 19–24: A function f and a rectangle $[a,b] \times [c,d]$ are given. Plot the real and imaginary parts of the function over $\{x + iy : a \le x \le b, c \le y \le d\}$ and locate any singularities in the rectangle.

19. $1/(z^3 - z^2 + z - 1)$, $[-2,2] \times [-2,2]$
20. $1/(z^4 - 2z^3 + 3z^2 - 2z + 2)$, $[-2,2] \times [-2,2]$
21. $\tan(\pi z)$, $[-2,2] \times [-2,2]$
22. $\tanh(\pi z)$, $[-2,2] \times [-2,2]$
23. arcsin, $[-5,5] \times [-2,2]$
24. arctan, $[-2,2] \times [-5,5]$

25. The Runge phenomenon is related to the notion of radius of convergence for Taylor series.
    (a) Find the radius of convergence for the Maclaurin series for $\arctan x$.
    (b) Find the smallest Runge curve centered at the origin that intersects singularities of $\arctan z$. Estimate the points where the Runge curve intersects the real axis to two significant digits.
    (c) If you attempted to approximate $\arctan x$ on larger and larger intervals $[-a, a]$ with a Taylor polynomial or by interpolation on a uniform grid, would divergence of the Taylor series or the Runge phenomenon become a problem first? *Note.* This says nothing of the Lebesgue constant, which is also a concern.

26. Let Λ_m be the Lebesgue constant for a uniform grid of degree m. Prove

$$\lim_{m \to \infty} \Lambda_m^{1/m} = 2.$$

(Hence, Λ_m is on the order of 2^m for large m.) *Hint.* Find and prove an upper bound analogous to (9.1) and then show that the mth roots of the upper and lower bounds both converge to 2 as $m \to \infty$.

27. The gamma function $\Gamma(x)$ is an extension of the factorial function from nonnegative integers to real and complex numbers. Using the identity $\Gamma(x+1) = x\Gamma(x)$, which holds for $x \neq 0, -1, -2, \ldots$, prove that the Lagrange basis polynomial in the proof of Theorem 9.4 satisfies

$$l_i(x) = (-1)^m \frac{x\Gamma(m+1-x)}{\Gamma(1-x)} \frac{1}{x-i}$$

for $x \neq i$.

28. This exercise considers the barycentric weights for a uniform grid.
    (a) For a given degree m, which barycentric weights have the largest magnitude? Which have the smallest magnitude? For a uniform grid of degree 20, what is the ratio between these weights?
    (b) Let $p(x)$ be an interpolating polynomial defined by $p(x_i) = y_i$, $i = 0, \ldots, 20$. Suppose a single sample value y_i were perturbed to $\hat{y}_i$. Based on the barycentric weights, a perturbation in which sample value would have the most drastic effect on the entire interpolant? Which would produce the least change? Explain why, and then demonstrate the effect graphically.